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On the Stable Limit-point of a Modified Van Der Pol Equation

[1]S.B. Tiwari, [1]B. Nageswara Rao, [2]H.R. Nataraja and [3]K.S. Sai
[1]Structural Analysis and Testing Group, Vikram Sarabhai Space Centre, Trivandrum-695022, India
[2]Compressor Group, Gas Turbine Research Establishment, C.V. Raman Nagar, Bangalore-560093, India
[3]Department of Mathematics, S.V.H. College of Engineering, Machilipatnam-521002, India

Abstract: This research is concerned primarily with nonlinear oscillations of a modified van der Pol equation. The motion is represented by the harmonic oscillator equation, with the addition of a small nonlinear term. The governing differential equation falls under autonomous category. The solution of the problem is examined utilizing the method of slowly varying amplitude and phase (the Krylov-Bogoliubov-Mitropolsky technique). Stationary values of the amplitude are obtained and discussed their stability. It is noted that the stable limit-cycles of the differential equation in the higher order averaging method can be identified easily from the time derivative of the amplitude function and the sign of its derivative at the stationary value of the amplitude.

Key words: van der pol equation, amplitude, phase, limit-cycles, KBM technique

Introduction

Most vibrations are nonlinear and transient in nature, occur suddenly and randomly, have varying magnitudes and are difficult to analyze by linearizing and applying standard methods, as each case differs from the other. It is now possible to study nonlinear vibrations analytically using differential equations due to Duffing, van der Pol, Mathieu, etc., or graphically using integral or response curves or phase-portraits (Stoker, 1966; Nayfeh and Mook, 1979; Mickens, 1981; Thompson and Steward, 1989; Srinivasan, 1995). Interesting studies were made on nonlinear oscillations from the solution of equations of motion related to undamped free vibrations, forced vibrations, damped free vibrations and damped forced vibrations. These are characterized by the frequency and amplitude of oscillations. Rao (1992), Sarma *et al*. (1995, 1997a, b), Sarma and Rao (1998), Potti *et al*. (1999), Swamy *et al*. (2003), Muthurajan *et al*. (2005) and Tiwari *et al*. (2005) have obtained solutions of the equations of motion of a conservative system by several methods.

The problem of the determination of limit-cycles is fundamental in the theory of oscillations of nonlinear non-conservative systems. The limit-cycle is a closed integral curve in the phase-plane, which corresponds to a periodic solution of the equation of motion. It has the important that all integral curves in its neighborhood spiral toward it from both outside and inside. The problem can be solved by direct methods only in a few cases. It is a very difficult task to identify the presence of limit-cycles for a given differential equation.

Burnette and Mickens (1996) have examined the stability of limit-cycles on a modified van der Pol equation and proposed a criterion for identification of the stationary values of the amplitude corresponding to the actual limit-cycles of a general nonlinear differential equation, which is

Corresponding Author: Dr. B. Nageswara Rao, Structural Analysis and Testing Group, Vikram Sarabhai Space Centre, Trivandrum-695022, India Tel: +91-471-2565831 Fax: +91-471-2564181

represented by the harmonic oscillator equation, with addition of a small nonlinear term. This study demonstrates the identification of stable limit-cycles of the modified van der Pol equation in the higher order averaging method from the time derivative of the amplitude function. It also explains the inadequacy of the criterion as proposed by Burnette and Mickens (1996).

Analysis

Burnette and Mickens (1996) have considered a modified van der Pol equation:

$$\frac{d^2y}{dt^2}+y=\varepsilon\left\{-\beta y^3+\left(1-y^2\right)\frac{dy}{dt}\right\},\qquad 0<\varepsilon<<1,\qquad 0<\beta=O(1), \tag{1}$$

to provide a criterion for determining the actual limit-cycles that occur in a general non-linear differential equation:

$$\frac{d^2y}{dt^2}+y=\varepsilon F\left(y,\frac{dy}{dt}\right),\quad 0<\varepsilon<<1, \tag{2}$$

for which the averaging method of Krylov-Bogoliubov-Mitropolsky (KBM) can be applied. Here F is a polynomial function of its arguments.

According to the generalized method of KBM, the nth order approximation to the solution of Eq. 2 is (Mickens, 1981).

$$y=a\cos(\psi)\ +\sum_{i=1}^{n}\varepsilon^i\,u_i(a,\psi)\,, \tag{3}$$

Where $u_1, u_2, \ldots\ldots\, u_{n-1}$ are periodic functions of Ψ with a period $2\,\pi$. The quantities a (t) and $\psi_{(t0}$ are defined by

$$\frac{da}{dt}=K(a)=\sum_{i=1}^{n}\varepsilon^i\,A_i(a) \tag{4}$$

$$\frac{d\psi}{dt}=1+\sum_{i=1}^{n}\varepsilon^i\,B_i(a) \tag{5}$$

Here K (a) is the time derivative of the amplitude function.

Applying the second approximation (n = 2) of the KBM averaging method to Eq. 1, Burnette and Mickens (1996) have obtained the amplitude Eq. 4, which is rewritten (after defining $z = a^2$) in the form:

$$\frac{dz}{dt}=\bar{A}(z) \tag{6}$$

For this case, $\bar{A}$ (z) is a cubic polynomial and one of the stationary values of the amplitude is zero, which is reported as the unstable fixed point. They have obtained other two stationary values of the amplitude by solving a quadratic equation and identified them as a stable and unstable limit points. The behavior of these values was also examined from a plot showing the variation of $\bar{A}$ (z) with z. Stationary amplitudes can be found from the points of intersection of the curve $\bar{A}$ (z) with the z-axis.

Though not explained clearly by Burnette and Mickens (1996), one can guess from the plot that stable amplitudes correspond to the points where the curve intersects the z-axis from the upper side and unstable amplitudes correspond to the points where the curve intersects the z-axis from the lower side. This phenomenon can also be explained from the positive and negative values of $d\bar{A}/dz$ at the stationary values of the amplitude. Stable stationary values of the amplitude are those at which $d\bar{A}/dz < 0$, whereas unstable stationary values of the amplitude are those at which $d\bar{A}/dz > 0$.

Burnette and Mickens (1996) have observed that the stationary values of the amplitude corresponding to the unstable limit point increases without bound as $\varepsilon \to 0$ (referred this to the spurious limit-cycle), whereas the stationary values of the amplitude corresponding to the stable limit point is bounded as $\varepsilon \to 0$ (referred this to the actual limit-cycle). Based on this observation, they proposed a criterion for the determination of the actual limit-cycles in the use of higher order averaging techniques. In that criterion, the actual limit-cycles correspond to solutions of K (a) = 0 in Eq. 4 that are bounded as $\varepsilon \to 0$. And suggested to ignore all other solutions correspond to spurious limit-cycles, in the analysis of the properties of the solutions to Eq. 1. However, usage of this procedure to a general non-liner differential Eq. 2 requires expressions for the stationary values of the amplitude in terms of ε for applying the limit condition $\varepsilon \to 0$. The task is involved, if one seeks a solution for Eq. 2 applying the higher order averaging method of KBM.

It is very interesting to note that the cubic Eq. 11 of Burnette and Mickens (1996) reduces to a quadratic equation, if one applies the limiting condition as $\varepsilon \to 0$ and the stationary values of the amplitude correspond to the first order approximation of KBM averaging method. That is the reason why one of the stationary values of the amplitude from the cubic Eq. 11 of Burnette and Mickens (1996) becomes infinity, when $\varepsilon = 0$. In general, this criterion may not be convenient for identification of the stationary values of the amplitude corresponding to the actual limit-cycles of a general non-linear differential Eq. 2 through higher order averaging methods of KBM.

In second order approximation (n = 2), the functions in Eq. 3 and 5 are obtained as

$$u_1(a,\psi)=\frac{a^3}{32}(\beta\cos(3\psi)-\sin(3\psi)), \tag{7}$$

$$A_1(a)=\frac{a}{2}\left(1-\frac{a^2}{4}\right), \tag{8}$$

$$A_2(a)=-\frac{\beta a^3}{16}\left(3-\frac{a^2}{2}\right) \tag{9}$$

$$B_1(a)=\frac{3}{8}\beta a^2 \tag{10}$$

$$B_2(a)=-\frac{1}{8}+\frac{a^2}{8}-\frac{(7+15\beta^2)}{256}a^4 \tag{11}$$

The amplitude relation (9) of Burnette and Mickens (1996) needs correction due to their erroneous expression for $A_2(a)$.

Defining $z = a^2$ and using Eq. 8 and 9 in Eq. 4, one obtains

$$\frac{dz}{dt}=\bar{A}(z)=\varepsilon z\left\{1-\left(1+\frac{3\varepsilon\beta}{2}\right)\frac{z}{4}+\frac{\varepsilon\beta z^2}{16}\right\}=\varepsilon z(1-\alpha_1 z)(1-\alpha_2 z), \tag{12}$$

Where,

$$\alpha_1 + \alpha_2 = \frac{1}{4}\left(1 + \frac{3\varepsilon\beta}{2}\right);$$

$$\alpha_1 \, \alpha_2 = \frac{\varepsilon\beta}{16}; \text{ and}$$

$$\alpha_1 - \alpha_2 = \frac{1}{4}\sqrt{\left(1 - \frac{\varepsilon\beta}{2}\right)^2 + 2\varepsilon^2\beta^2} > 0.$$

It should be noted that α_1 and α_2 are greater than zero for $\varepsilon \neq 0$. Hence, the stationary values of the amplitude can be obtained from the three roots of $\bar{A}$ (z):

$$z_1 = 0; \; z_2 = \frac{1}{\alpha_1}; \quad \text{and} \quad z_3 = \frac{1}{\alpha_2}.$$

For the case $\varepsilon = 0$,

$$\alpha_1 + \alpha_2 = \frac{1}{4}, \alpha_1 \, \alpha_2 = 0 \text{ and } \alpha_1 - \alpha_2 = \frac{1}{4},$$

Which imply that

$$\alpha_1 = \frac{1}{4} \text{ and } \alpha_2 = 0.$$

This corresponds to:

$$z_2 = 4 \text{ and } z_3 = \infty.$$

For a negligibly small ε, α_2 in Eq. 12 becomes insignificant and the third stationary value of the amplitude will be extremely large, whereas the second stationary value of the amplitude will be close to that obtained from the first approximation of the averaging method.

Identification of stable limit-point for Eq. 1 is done here based on the positive and negative values of $d\bar{A}/dz$ at the stationary values of the amplitude as follows:

$$\frac{d\bar{A}}{dz} = \varepsilon > 0 \qquad \text{(at } z_1 = 0\text{, unstable limit point)}$$

$$\frac{d\bar{A}}{dz} = -\frac{\varepsilon}{\alpha_1}(\alpha_1 - \alpha_2) < 0 \qquad \left(\text{at } z_2 = \frac{1}{\alpha_1}\text{, stable limit-point}\right)$$

$$\frac{d\bar{A}}{dz} = \frac{\varepsilon}{\alpha_2}(\alpha_1 - \alpha_2) > 0 \qquad \left(\text{at } z_3 = \frac{1}{\alpha_2}\text{, unstable limit-point}\right)$$

This leads to the conclusion that Eq. 1 has a single stable limit cycle for the second approximation of the averaging method of KBM. For sufficiently small value of ε, the first approximation of the averaging method will give the properties of the differential Eq. 1. The addition of terms of higher approximation does not modify the qualitative character of the solution, but merely modify their quantitative nature slightly.

Results and Discussion

To demonstrate the identification of stable limit-cycle of the modified van der Pol Eq. 1 from the time derivative of the amplitude function, the parameters in the differential equation are specified as:

$$\varepsilon = 0.1 \text{ and } \beta = 0 \text{ and } 10$$

Figure 1 and 2 show the variation of $\bar{A}$ (z) and $d\bar{A}/dz$ with z. when $\beta = 0$, the amplitude equation corresponding the second approximation reduces to that of first approximation, having two stationary v alues of the amplitude. It can be seen from Fig. 1 that $d\bar{A}/dz > 0$ at $z = 0$ and

Table 1: Variation of $d\bar{A}/dz$ with β at the stationary values of the amplitude, $a(\equiv\sqrt{z})$, for $\varepsilon = 0.1$ (one of the stationary amplitudes, $z_1 = 0$, $\left.\frac{d\bar{A}}{dz}\right|_{z=z_1} = \varepsilon = 0.1 > 0$)

β	z_2	$d\bar{A}/dz$ at $z = z_2$	z_3	$d\bar{A}/dz$ at $z = z_3$
0	4.0	-0.1	-	-
1	3.7906	-0.0910	42.21	1.014
2	3.5660	-0.0841	22.43	0.5291
3	3.3333	-0.0792	16.00	0.3800
4	3.1010	-0.0760	12.90	0.3160
5	2.8769	-0.0741	11.12	0.2866
6	2.6667	-0.0733	10.00	0.2750
7	2.4735	-0.0732	9.2408	0.2736
8	2.2984	-0.0736	8.7016	0.2786
9	2.1410	-0.0742	8.3034	0.2878
10	2.0	-0.075	8.0	0.3

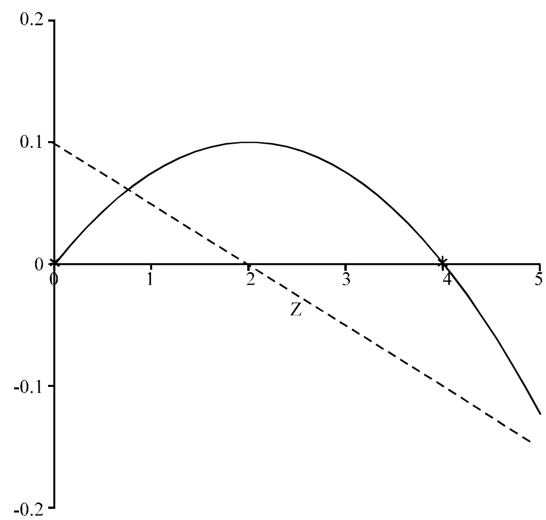

Fig. 1: Variation of $\bar{A}$ (z) and $d\bar{A}/dz$ with *z* for $\varepsilon = 0.1$ and $\beta = 0$. Solid line represents variation of $\bar{A}$ (z) whereas broken line represents variation of $d\bar{A}/dz$. Star refers to the squared value of the stationary amplitude

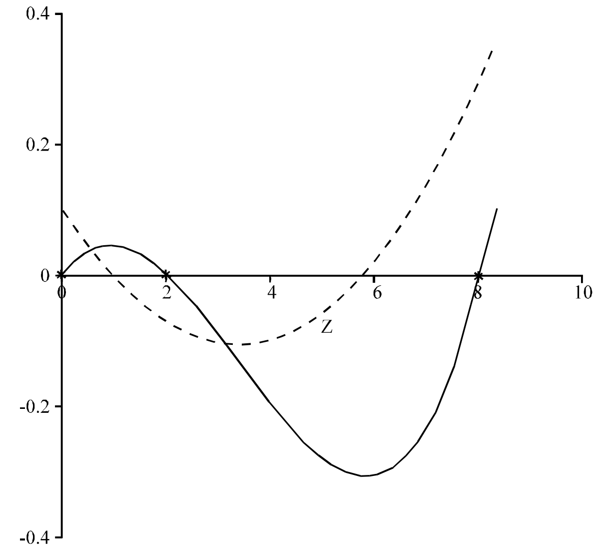

Fig. 2: Variation of $\bar{A}$ (z) and $d\bar{A}/dz$ with z for $\varepsilon = 0.1$ and $\beta = 0$. Solid line represents variation of $\bar{A}$ (z) whereas broken line represents variation of $d\bar{A}/dz$. Star refers to the squared value of the stationary amplitude

$d\bar{A}/dz < 0$ at $z = 4$, is the stable limit-point. When $\beta = 10$, as expected, $\bar{A}$ (z) curve meets the z-axis at three points. $d\bar{A}/dz$ at one of the stationary points of the amplitude is negative and hence it corresponds to the stable limit-point of differential Eq. 1. Table 1 gives the variation of $d\bar{A}/dz$ with β at the three stationary values of (z_1, z_2 and z_3) of the amplitude, $a\,(\equiv\sqrt{z})$. It is found that z_2 is the stable limit-point for all values of β (since, $\left.\frac{d\bar{A}}{dz}\right|_{z=z_2} < 0$), whereas, z_1 and z_3 are the unstable limit-points.

It can be seen from Fig. 1 and 2 that the stable stationary amplitudes correspond to the points where the $\bar{A}$ (z) curve intersects the z-axis from the upper side (at the point of intersection, $d\bar{A}/dz < 0$). The unstable stationary amplitudes correspond to the points where the $\bar{A}$ (z) curve intersects the z-axis from the lower side (at the point of intersection, $d\bar{A}/dz > 0$).

Conclusions

The stable limit-cycles of differential Eq. 1 in the higher order averaging method can be identified easily from the time derivative of the amplitude function and the sign of its derivative at the stationary values of the amplitude. Hence the new criterion of Burnette and Mickens (1996) may not possess any additional advantages in identifying the stable-limit point of a modified van der Pol equation.

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