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An Improved Linearized Semi-implicit Runge-Kutta Methods

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Abstract: We present a framework for the stability analysis of a linearized semi-implicit Runge-Kutta Method. To determine an A or L stable LSIRM. The conventional procedure has been to estimate the parameters from the construction and then test for A-or L-stability. It is now questioned whether conditions cannot be imposed on the parameters a priori to achieve the required stability. The general LSIRM of order two and three respectively were, therefore, constructed using Taylor's expansion. The stability function for each method was derived from the stability analysis; criteria were set for A-or L-stability.

Key words: Linearized methods, stability function, rosenbrock, complex plane

Introduction

Butchher (1963) gave the general R stage class of LSIRM for the initial value problem

$$\frac{dy}{dt} = f(y), y(t_0) = y_0 \tag{1}$$

as

$$\begin{aligned} y_{n+1} - y_n &= h\sum_{r=1}^{R} C_r K_r \\ K_1 &= f(y) + \alpha_1 h \frac{dy}{dt}(y_n)K_1 \end{aligned} \tag{2}$$

Two-Stable Order 2 Methods (The 2-2 Methods)

The 2-2 methods is given as

$$\begin{aligned} y_{n+1} - y_n &= h(C_1K_1 + C_2K_2) + 0(h^3) \\ K_1 &= f + \alpha_1 h(y_n)K \\ K_2 &= f(y_n + hb_{21}K_1) + \alpha_2 hf_y(y_n + h\beta_{21}K_1)K_2 \end{aligned} \tag{3}$$

$$\begin{aligned} &A_{22} = f(y_n + h\beta_{21}K_1) = f + h\beta_{21}f_yK_1 + 0(h^2) \\ &B_{22} = f_y + h\beta_{21}f_yK_1 + 0(h^2), A_{22}hB_{22} = \alpha_2 hf_y + 0(h^2), K_2 = A_{22} + \alpha_2 h\beta_{22}K_2 \\ &= f + h\beta_{21}f_yK_1 + \alpha_2 hf_yK_2 = f + h\beta_{21}f_y(f + \alpha_1 hff_y) + \alpha_2 hf_yK_2 = f + h\beta_{21}ff_y + \alpha_2 hf_yK_2 + 0(h^2) \end{aligned} \tag{4}$$

$$C_2K_2 = C_2f + C_2\beta_{21}ff_y + hC_2\alpha_2 ff_y$$

Using (3), $C_1K_1 + C_2K_2$ is now compared with
$\Phi(x, y; h) = f + hff_y + 0(h^2)$

Therefore, the NEQs for the above process are:

$$C_1+C_2 = 1$$
$$C_1\alpha_1+C_2\alpha_2+C_2\beta_{21} = {}^1/_2, \text{ With } \beta_{21} = 0$$

We have the equations in five unknowns. Fatunla (1988) and Iserles (1996). Thus, we are free to fix three parameters giving a three parameters family of solutions. For example, let: $C_1 = ¼$, $\alpha_1 = ½$, $\beta_{21} = -{}^1/_6$.

Then, solving the NEQs, we have:
$C_2 = ¾$, $\alpha_2 = {}^2/_3$, $\beta = 0$. The resulting method is:

$$y_{n+1}-y_n = h({}^1/_4K_1+{}^3/_4K_2)+O(h^3)$$

$$K_1 = f(y_n)+\frac{1}{-}h\frac{df}{dy}(y_n)K_1$$

$$K_2 = f(y_n-\frac{1}{6}hK_1)+\frac{2}{3}h\frac{df}{dy}(y_n)K_2 \tag{5}$$

(The above method is L-Stable and will be seen from theorem 2.

Rosenbrock (1963) gave the following values for the parameters of a 2-2 method, i.e.,

$$\alpha_1 = 1\frac{\sqrt{2}}{2}$$

$$\beta_{21} = \frac{\sqrt{2}-1}{2}$$

$$C_1 = 0,\ C_2 = 1,\ \beta_{21} = 0, \text{ which are seen to satisfy (5).} \tag{5a}$$

The Stability Function: $\mu_{22}(\lambda h)$

$$y_{n+1}-y_n = h(C_1K_1+C_2K_2)+O(h^3) \tag{6}$$

(The terms must have $\le h^2$ after multiplying K_i by C_ih or $\le h$, before multiplying by C_ih).

$$K_1 = f+\alpha_1hf_yK_1$$

Apply to the scalar test equation, then:

$$K_1 = \lambda y_n+\alpha_1h\lambda K_1, \text{ since } f_y(\lambda y_n) = \lambda$$

$$K_1 = \frac{\lambda y_n}{1-\alpha_1h\lambda}$$

$$K_2 = f(y_n+h\beta_{21}K_1)+\alpha_2h\ \frac{df}{dy}(y_n)K_2, \text{ (Since } \beta_{21} = 0\text{)}, K_2=\lambda(y_n+h\beta_{21}K_1)+\alpha_2\lambda hK_2$$

$$K_2(1-\alpha_1\lambda h) = \lambda(y_n+h\beta_{21}K_1)$$

$$= \lambda\left(y_n + \frac{h\beta_{21}\lambda y_n}{1-\alpha_1 h\lambda}\right)$$

$$K_2 = \left(\frac{\lambda - \alpha_1\lambda^2 h + \beta_{21}\lambda^2 h}{(1-\alpha_2 h\lambda)(1-\alpha_1 h\lambda}\right)y_n \qquad (7)$$

Stability Analysis

The method given by Rosenbrock (1963) and Okonta (2004) is said to be A-Stable. We prove that this is true.
For the method

$$\alpha_1 = 1-\frac{\sqrt{2}}{2}, \quad \alpha_1 = -\frac{\sqrt{2}}{2}$$

$$\beta_{21} = \frac{\sqrt{2}-1}{2}, \; C_1 = 0, \; C_2 = 1, \; B_{21} = 0$$

Substituting into (7),

$$A_1 = \sqrt{2}-1; \; A_2 = 2-\sqrt{2}$$

$$B_2 = {}^3\!/_2 - \sqrt{2}$$

$$\mu_{22}(z) = \frac{1+A_1Z+A_2Z^2}{1-B_1Z+B_2Z^2}$$

$$= \frac{1+(\sqrt{2}-1)Z}{1-(2-\sqrt{2}+({}^3\!/_2-\sqrt{Z})Z^2} \qquad (8)$$

A-Stability

By definition, the method is A-Stable if $|\mu_{22}(Z)|<1$ whenever $Re(Z) <0$. This reduces to

$$\left|1+(\sqrt{2}-1)Z\right| < \left|1-(2-\sqrt{2})Z+({}^3\!/_2-\sqrt{2})Z^2\right|$$

Take a test case of $Re(Z) = -1$
Then

$$\left|1+(\sqrt{2}-1)\right| < 1+(2-\sqrt{2})+({}^3\!/_2-\sqrt{2})$$

$$\left|2-\sqrt{2}\right| < \left|4\,{}^1\!/_2-(2\sqrt{2}\right|$$

|0.5859|<|1.6715|which is true. Hence, the method is confirmed to be A-Stable.

L-Stability

By definition, the method is L-Stable if:

- it is A-Stable and

- $\lim_{Re(z)\to -\infty} |\mu(z)|$

$$\mu_{22}(Z) = \frac{1+\sqrt{2}-1)Z}{1-(2-\sqrt{2})Z+(3/_2-\sqrt{Z})Z^2}$$

$$\lim_{Re(z)\to -\infty} |\mu(z)|$$

$$= \text{Lt} \qquad \frac{\sqrt{2}-1}{\sqrt{2}-2+(3-2\sqrt{2})Z} = 0$$

$$\lim_{Re(z)\to -\infty} |\mu(z)|$$

By L'Hospital's Rule, the method is stable.

Poles Lemma

Let r be an arbitrary rational function which is not a constant. Then, $|r(Z)| < 1$ for all $Z \in C$, where C is the complex plane, if and only if all the poles of r have positive real parts and $|r(it) \leq 1$ for all $t \in R$.

Theorem 2.2

A two stage 2nd order LSIRM is A-stable if

- $\alpha_1 > 0$; $\alpha_2 > 0$ and
- $A_1^2 + A_2^2 = B_1^2 + B_2^2 - 2B_2$

where

$$A_1 = 1-(\alpha_1+\alpha_2),\ A_2 = \alpha_1\alpha_2 + C_1\alpha_2 - C_2\alpha_1 + C_2\beta_{21},\ B_1 = \alpha_1+\alpha_2,\ B_2 = \alpha_1\alpha_2$$

Proof

From the poles Lemma, the stability function $\mu_{22}(Z)$ (8) will have poles at the points where $B_2Z^2 - B_1Z + 1 = 0$

$$Z = \frac{\alpha_1\alpha_2 \pm \sqrt{(\alpha_1+\alpha_2)-4\alpha_1\alpha_2}}{2\alpha_1\alpha_2} \qquad (9)$$

The real roots are therefore $= 1/\alpha_1$, $1/\alpha_2$ These will be positive only if $1/\alpha_1 > 0$ and

$1/\alpha_2 > 0$ or $\alpha_1 > 0$ and α_2

Also by the poles lemma, we expect $|\mu(it)| \leq 1$ for all $t \in R$.

i.e., $\frac{1 + A_1(it) + A_2(it)^2}{1 - B_1(it) + B_2(it)^2}$

$$|1 + A_1(it) + A_2(it)^2|^2 \leq |\ 1 - B_1(it) + B_2(it)^2\ |^2$$

$$A_1^2+A_2^2t^2-2A_2 = B_1^2+B_2t^2-2B_2 \qquad (10)$$

If the parameters of the method are not given then, A_1, A_2, B_1, B_2 may be found from the stability function.

$\mu(Z) = \frac{1 + A_1Z + A_2Z^2}{1 - B_1Z + B_2Z^2}$, It is easily seen from $\mu_{22}(Z)$ that

$$\alpha_1 = 1 - \frac{\sqrt{2}}{2}$$

$\alpha_2 = 1 - \frac{\sqrt{2}}{2} > 0$, Which satisfies condition (i) of the theorem.

Also, from
$B_{21} = 3/2 - \sqrt{2}$
obtained by application of (5).
It is seen that

$$A_1^2 + A_2^2 - 2A_2 = 3-2\sqrt{2} = 0.1715$$
$$B_1^2 + B_2^2 - 2B_2 = 7\,1/4 - 5\sqrt{2} = 0.1715$$
$$\text{i.e., } A_1^2 + A_2^2 - 2A_2 \le B_1^2 + B_2^2 - 2B_2$$

which altogether prove that the method is A-Stable.

Having known the conditions on the parameters that will lead to A-Stability, it becomes necessary to examine the conditions on the parameters that may guarantee L-Stability.

This leads us to propose the following theorem.

Theorem

Given the parameters α_1, α_2, C_1, C_2 and β_{21} in a 2-stage second order Rosenbrock method, a condition for L-stability is that:

$$B_{21} = \alpha_1 - \frac{\alpha_1}{C_2}(\alpha_1 + C_1) \qquad \text{Provided } C_2 \neq 0$$

Proof

The stability function is

$$\mu_{22}(Z) = \frac{1 + A_1Z + A_2Z^2}{1 - B_1Z + B_2Z^2}$$

Since $\beta_2>0$ by theorem 2.1 putting $A_2 = 0$ and find the limit of $\mu_{22}(Z)$ as $Re(Z) \to -\infty$
The limit

$$\left|\frac{1 + A_1Z}{1 - B_1Z + B_2Z^2}\right|$$

as $Re(Z) \to -\infty$

$$= \underset{\mathrm{Re}(Z)\to-\infty}{\mathrm{Limit}} \left| \frac{A_1}{-B_1+2B_2Z} \right| = 0$$

By L'Hospitals Rule, since A_1, B_1 and B_2 are constants. Therefore, $A_2 = 0$ produces the required limit of μ_{22} as Re(Z) →-∞ but from (7)

$$A_2 = \alpha_1\alpha_2+C_1\alpha_2-C_2\,\alpha_1+C2\beta_{21}$$

Therefore, if $A_2 = 0$

$$\alpha_1\alpha_2+C_1\alpha_2-C2\,\alpha_1+C2\beta_{21} = 0$$

$$\text{i.e., } \beta_{21} = \frac{C_2\alpha_1 - \alpha_1\alpha_2 - C_1\alpha_2}{1-B_1Z+B_2Z^2}$$

$$\beta_{21} = \alpha_1 - \frac{\alpha_1}{\alpha_2}(\alpha_1 + C_1) \text{ provided } C2 \neq 0$$

(Verification shows that (3.2) is L-stable)
The parameter of the method is $\mu_{22}(Z)$.

Two-Stage, Order 3 Methods (The 2-3 Method)
Construction

$$y_{n+1} = h(C_1K_1+C_2K_2)+O(h^4)$$

$$K_1{}^2 = f^2+2\alpha_1hf^2fy+3\,\alpha_1{}^2h^2f^2fy^2+O(h^3)$$

$$K_2 = f(y_n+h\beta_{21}K_1)+\alpha_2\frac{df}{dy}y_n+h\beta_{21}K_1)K_2 \tag{11}$$

$$= A_{23}+\alpha_1h\beta_{23}K_2,\ \text{Now,}A_{23} = f(y_n+h\beta_{21}K_1) \tag{12}$$

$$= f+h\beta_{21}K_1fy+h^2/2\beta_{21}{}^2K_1{}^2fyy+O(h^3) \tag{13}$$

Similarly, $B_{23} = f_y(y_n+h\beta_{21}K_1)$

$$= f_y+h\beta_{21}K_1f_{yy}+h^2/_2\beta_2{}^2K_1{}^2f_{yyy}+O\,(h^3) \tag{14}$$

In (12) we shall need A_{23}, B_{23} and $B_{23}{}^2$. By inspection of (12), we observe that the factors α_2h and $\alpha_2{}^2h^2$ permit us to discard O (h^2) in the expansion of A_{23}, B_{23} and discard O(h) in the expansion of $B_{23}{}^2$ and K_2.

Hence, using (13) and (14),

$$A_{23}B_{23} = ffy+h\beta_{21}K_1ffyy+h\beta_{21}K_1fy^2 \tag{15}$$

$$B_{23}{}^2 = f^2y \tag{16}$$

From (12) also using (13), (15) and (16), Therefore,

$$K_2 = f{+}h(\beta_{21}{+}\alpha)ff_y{+}h^2(\tfrac{1}{2}\beta_{21}{}^2{+}\alpha_2h^2\beta_{21})f^2f_{yy}{+}h^2(\alpha_1\beta_{21}{+}\alpha_2\beta_{21}{+}\alpha_2{}^2)ff^2{}_y \quad (17)$$

From (11)

$$C_1K_1{+}C_2K_2 = f(C_1{+}C_2){+}h(C_2\beta_{21}{+}C_2\alpha_2{+}C2\alpha_2)ffy{+}h^2(\tfrac{1}{2}C_2\beta_{21}{}^2{+}C_2\alpha_2\beta_{21}{+}C_2\alpha_2\beta_{21})ff^2{}_y \quad (18)$$

Comparing with

$$\Phi_T = f{+}h\tfrac{1}{2}ff_y{+}h^2({}^1/_6f^2fyy{+}{}^1/_6ff^2y)$$

Then, the NEQs are:

$$\begin{aligned} &C_1{+}C_2 = 1\\ &C_1\alpha_1{+}C_2\alpha_2{+}C_2\beta_{21}{=}\ \tfrac{1}{2}\\ &\tfrac{1}{2}C_2\beta_{21}{}^2{+}C_2\beta_{21} = {}^1/_6\\ &C_1{}^2\alpha_1{}^2{+}C_2\alpha_2{}^2{+}(\alpha_1{+}\alpha_2)C_2\beta_{21} = {}^1/_6 \end{aligned} \quad (19)$$

The exact replica of these equations is found in Reosenbrock (1963) and Lambert (1977). The following values of the parameters are in perfect agreement with (12) and are said to produce an A-stable 2-3 Method.

These are:

$$\begin{aligned} &C_1 =-0.413,\ 154,\ 32\\ &C_2 = 1.413,\ 154,\ 32\\ &\alpha_1 = 1.408,\ 248,\ 29\\ &\alpha_2 = 0.591,\ 751,\ 71\\ &b_{21} = 0.173,\ 786,\ 67\\ &\beta_{21} = 0.173,\ 786,\ 67 \end{aligned} \quad (20)$$

The set (20) has four equations in six unknowns. So, we have 2 free parameters producing a 2 parameter family of solution.

Stability Function $\mu_{22}(\lambda h)$

Applying to the scalar test equation,

$$\frac{K_1 = \lambda y_n + \alpha_1\lambda^2hy_n}{1-\alpha_1{}^2(\lambda h)^2} \quad (21)$$

From (17)

$K_1 =\ f{+}hb_{21}K_1fy{+}h^2/2b_{21}{}^2K_1{}^2fyy{+}\alpha_2hffy{+}h^2\alpha_2\beta_{21}K_1ffyy{+}h^2\alpha_2b_{21}K_1f^2y{+}\alpha_2{}^2h^2f^2yK$. Hence,

$K_2 = \lambda y_n{+}hb_{21}K_1\lambda{+}h^2/2b_{21}{}^2K_1{}^2(0){+}\alpha_2h\lambda y_n\lambda{+}h^2\alpha_2\beta_{21}K_1\lambda y_n(0){+}h^2\alpha_2b_{21}K_1\lambda^2{+}\alpha_2{}^2h^2\lambda^2K_2$

$$= \lambda y_n{+}\alpha_2\lambda^2hy_n{+}\frac{b_{21}\lambda h +(\lambda + \alpha_1\lambda^2h)y_n + \alpha_2b_{21}\lambda^2h^2(\lambda + \alpha_1\lambda^2h)}{1-\alpha_1{}^2(\lambda h)^2}$$

$$= \lambda y_n + \alpha_2\lambda^2 hy_n + \frac{b_{21}\lambda^2 h + b_{21}\alpha_1\lambda^3 h^2 y_n}{1-\alpha_1{}^2(\lambda h)^2} + \frac{\alpha_1 b_{21}\lambda^3 h^2 + \alpha_1\alpha_2 b_{21}\lambda^4 h^3)y_n}{1-\alpha_1{}^2(\lambda h)^2}$$

$$= \frac{Q}{1-\alpha_1{}^2(\lambda h)^2} \quad \text{Where}$$

$$Q = \lambda\text{-}\alpha_1{}^2\lambda^3 h^2 + \alpha_2\lambda^2 h\text{-}\alpha_1{}^2\alpha_2\lambda^4 h^3 + b_{21}\lambda_2 h + b_{21}\lambda^3 h^2 + \alpha^2 b_{21}\lambda^3 h^2 + \alpha_1\alpha_2 b_{21}\lambda^4 h^3 \tag{22}$$

$$= \frac{Q}{(1-\alpha_1{}^2(\lambda h)^2)\,(1-\alpha_2{}^2(\lambda h)^2)y_n}$$

$$C_2 hK_2 = \frac{C_2 hQ}{(1-\alpha_1{}^2(\lambda h)^2)\,(1-\alpha_2{}^2(\lambda h)^2)y_n} \tag{23}$$

$$C_2 hK_2 = \frac{C_2\lambda h + C_1\alpha_1(\lambda h)^2}{(1-\alpha_1{}^2(\lambda h)^2} \tag{24}$$

The method (23) from Lambert (1977), is to be investigated for the purpose of demonstration we approximate the given values of the parameters to two decimal places.

$$C_1 = -0.41,\ C_2 = 1.41,\ \alpha_1 = 1.41,\ \alpha_2 = 0.59,\ b_{21} = 0.17,\ \beta_{21} = 0.17$$

A-Stability

A-Stability requires that

$$|\mu_{22}(Z)|<1 \text{ whenever } Re(Z)<0$$

i.e., $|1\text{-}Z\text{-}1.43Z^2\text{-}2.18Z^3\text{-}0.56Z^4|<|1\text{-}2.34Z^2+0.69Z^4$

For any $Re(Z) < 0$

Taken the test case, $Re(Z) = -1$. Then,

$$|0.19|<|0.65|$$

$0.19 < 0.65$, which is true. For $Re(Z) = -2$, $|1.76|<|2.68|$, etc.
Hence the method is A-Stable as asserted by Lambert (1977).

L-Stability

The requirement is that

Lt $|\mu_{23}(Z)|\rightarrow 0$ as $Re(Z)\rightarrow -\infty$

$$|\mu 23(Z)| = \frac{1+Z-1.43Z^2-2.18Z^3-0.56Z^4}{1-2.34Z^2+0.69Z^4}$$

By L'Hospitals rule, this reduces to 0.8 as $Re(Z)\rightarrow -\infty$

This is exactly the same limit obtained by Rosenbrock (1963), for the same method using the function $\psi(t)$ defined in the literature as an approximation to $e^{\lambda h}$. Since it $|\mu_{23}(Z)|\neq 0$ as $Re(Z)\rightarrow -\infty$. Hence, the method is not L-stable

Three Stage Order 3 Methods (The 3-3 Methods) Construction

$$y_{n+1}-y_n = h(C_1K_1+C_2K_2+C_3K_3)+O(h)^4$$

$$\text{But } K = f+h(b_{21}+\alpha_2)ffy+h^2(y^2b^2_{12}+\alpha_2\beta^{21})f^2fyy+h^2(\alpha_1b_{21}+\alpha_2b_{21}+\alpha_2^{\,2})ffy^2$$
$$\frac{df}{dy}y_n + h\beta_{31}K_1 + h\beta_{32}K_2)$$

Then,
$$K_1 = A_{33}+\alpha_3h\beta_{33}(A_{33}+\alpha_3h\beta_{33}K_3)$$
$$= A_{33}+\alpha_{3h}\beta_{33}A_{33}+\alpha_3^{\,2}h^2\beta_{33}^{\,2}K_3+O(h^3) \tag{25}$$

$$\begin{aligned} A_{33} &= f(y_n+hb_{31}K_1+hb_{32}K_2) \\ &= f+h(b_{31}K_1+b_{32}K_2)f_y+h^2/2(b_{31}K_1+b_{32}K_2)^2f_{yy}+O(h^3) \\ &= f+hb_{31}K_1f_y+b_{32}K_2f_y+h^2/2b_{31}^{\,2}K_1f_y+h^2b_{31}b_{32}K_1K_2f_{yy}+h^2/2b_{32}^{\,2}K_2^{\,2}f_{yy} \end{aligned} \tag{26}$$

In view of the terms h and h2 in (26), we have:

$$K_1 = f+\alpha_1hff_y+O(h^2)$$
$$K_1^{\,2} = f^2+O(h)$$
$$K_{22} = f^2+O(h)$$

$K_1K_2 = f^2+O(h)$. Therefore:

$$A_{33} = f+hb_{31}(f+\alpha_1hff_y)f_y+hb_{32}(f+hb_{21}ff_y+h\alpha_2ff_y)f_y+h^2/2b_{31}^{\,2}f^2f_{yy}+h^2b_{31}b_{32}f^2f_{yy}+h^2/2b_{32}^{\,2}f^2f_{yy} \tag{27}$$

$$B_{33} = fy(y_n+h\beta_{31}K_1+h\beta_{32}K_2)$$

$$= f_y+h\beta_{31}k_1f_{yy}+h\beta_{32}K_2f_{yy}+h^2/2\beta_{31}^{\,2}K_1^{\,2}f_{yyy}+h^2\beta_{32}K_1K_2f_{yyy}+h^2/2\beta_{32}^{\,2}K_2^{\,2}f_{yyy} \tag{28}$$

Using K_1 values in (26), we readily obtain:

$$\begin{aligned} B_{33} = \; & f_y+h\beta_{31}ff_{yy}+h^2\alpha_1\beta_{31}ff_yf_{yy}+h\beta_{32}ff_y+h^2\beta_{32}b_{21}ff_yf_{yy}+h^2\beta_{32}\alpha_2ff_yf_{yy} \\ & +h^2/2\beta_{31}^{\,2}f^2f_{yyy}+h^2/2\beta_{21}\beta_{31}f^2f_{yyy}+h^2/2\beta_{32}f^2f_{yyy} \end{aligned}$$

In view of the terms α_3h premultiplying $A_{33}B_{33}$ in (26) we have

$$A_{33}B_{33} = ff_y+hb_{31}ff^2_{\,y}+hb_{32}ff^2_{\,y}+h\beta_{31}f^2f_{yy}+h\beta_{32}f^2f_{yy}$$

$$\alpha_3h\beta_{33}A_{33} = \alpha_3hff_y+h^2\alpha_3b31ff^2_{\,y}+h^2\alpha_3b_{32}ff^2_{\,y}+h^2\alpha_3h\beta_{31}f^2f_{yy}+h2\alpha_3\beta_{32}f^2f_{yy} \tag{29}$$

$$\alpha_3^{\,2}h^2\beta_{33}^{\,2}K_3 = h^2\alpha_3^{\,2}ff^2_{\,y} \tag{30}$$

But $K_3 = A_{33}+\alpha_2 hB_{33}A_{33}+\alpha_2^2h^2B_{33}^2K_3$

So, we can write

$$C_3K_3 = C_3f+hC_3b_{31}ff_y+h^2C_3b_{31}\alpha_1ff^2{}_y+hC_3b_{32}ff_y+h^2C_3b_{32}b_{31}ff^2{}_y+ h^2C_3b_{32}\alpha_2ff^2{}_y+h^2/2C_3b_{31}{}^2f^2f_{yy}+h^2C_3b_{31}f^2f_{yy}+h^2/2C_3b_{32}{}^2f^2f_{yy}+h\alpha_3C_3ff_y+ h^2C_3\alpha_3b_{31}ff^2{}_y+h^2C_3\alpha_3b_{32}ff^2{}_y+h^2C_3\alpha_3\beta_{31}f^2f_{yy}+h^2C_3\alpha_3\beta_{32}f^2f_{yy}+h^2C_3\alpha_3{}^2ff^2{}_y$$

i.e.,

$$C_3K_3 = C_3f+h(C_3b_{31}+C_3b_{32}+\alpha_3C_3)ffy+h^2(½C_3b_{31}{}^2+C_3b_{31}b_{32}+½C_3b_{32}{}^2+C_3\alpha_3\beta_{31}+ C_3\alpha_3\beta_{32})f^2fyy+h^2(C_3b_{31}\alpha_1+C_3b_{32}b_{31}+C_3b_{32}\alpha_2+C_3\alpha_3b_{31}+C_3\alpha_3b_{32}+C_3\alpha_3{}^2)ff^2{}_y \quad (31)$$

Furthermore we write

$$C_1K_1+C_2K_2+C_3K_3 = Cf+hN_1ff_y+h^2N_2f^2f_{yy}+h^2N_3ff^2{}_y$$

Comparing with

$$\Phi_T = f+h/2ff_y+h^2/6f^2f_{yy}+h^2/_6ff^2{}_y, \text{ we have}$$

$$C_1 = 1,\ N_1 = ½,\ N_2 = {}^1/_6,\ N_3 = {}^1/_6$$

Where the normal equations now become:

$$C = C_1+C_2+C_3 = 1$$

$$N_1 = C_1\alpha_1+C_2\alpha_2+C_3\alpha_3+C_1b_{21}+C_3b_{31}+C_3b_{32} = ½$$

$$N_2 = ½(C_2b_{21}{}^2+C_3b_{31}{}^2+C_3b_{32}{}^2)+C_2\alpha_2\beta_{21}+C_3\alpha_3\beta_{31}+C_3\alpha_3\beta_{32}+C_3b_{31}b_{32} = {}^1/_6$$

$$N_3 = C_1\alpha_1{}^2+C_2\alpha_2{}^2+C_3\alpha_3{}^2+C_2\alpha_1b_{21}+C^2\alpha_2b_{21}+C_3\alpha_3b_{31}+C_3\alpha_3b_{32}+C_3b_{31}\alpha_1+C_3b_{32}\alpha_2+C_3b_{32}B_{31} = {}^1/_6$$

These are four equations in 1 unknowns which yield 8 free parameters and can therefore have 8 parameter family of solutions.

Stability Function $\mu_{33}(\lambda h)$

Applying K_1 and K_2 of order 3 to the test equation, we obtain

$$C_1hK_1 = \frac{C_2\lambda h + C_1\alpha_1(\lambda h)^2}{1-\alpha_1{}^2(\lambda h)^2}$$

and

$$C_2hK_2 = \frac{C_2hQ}{(1-\alpha_1{}^2)(\lambda h)^2(1-\alpha_1{}^2(\lambda h)^2)}$$

But $K_3 = A_{33}+\alpha_3hA_{33}B_{33}+\alpha_3{}^2h^2B_{33}{}^2K_3$

Applying to the scalar test equation, we find that:

$(1-\alpha_3{}^2(\lambda h)^2)K_3 = T(A_{33}+\alpha_2hA_{33}B_{33})$

Table 1: Global Error of y1 from yi* for each method at t = 0

Method	Solution y_i	Error $\|y_i-y_i^*\|$
RO: 2-2	0.83517031	0.154879
L-Stable	-0.15488264	0.154882642
RO: 2-3	0.33377213	0.65627767
A-Stable	-0.656277662	0.56277664
I.E. Scheme	1.037718056	0.04766826
L-Stable	0.047619047	0.047619045

or

$$K_3 = \frac{T(A_{33} + \alpha_2 h A_{33} B_{33})}{1 - \alpha_3^2 (\lambda h)^2} \tag{32}$$

where T means applying the numerator to the scalar test equation.

We can readily obtain

$$\mu_{33}(Z)\ \frac{y_{n+1}}{y_n} = 1 + C_1 hK_1 + C_2 hK_2 + C_3 hK_3$$

This yields

$$\mu_{33}(Z)\ \frac{1 + A_1 Z + A_2 Z^2 + A_3 Z^3 + A_4 Z^4 + A_5 Z^5 + A_6 Z^6}{1 - B_2 Z^2 + B_4 Z^4 - B_6 Z^6} \tag{33}$$

Numerical Experiments

In this section, we apply the methods discussed in this paper to a particular differential system on a comparative basis.

Problem

$$\frac{dx}{dy} = -0.1x - 99.9z$$

$$\frac{dx}{dy} = -200Z - 99.9Z, x(0) = 2,\ Z(0) = 1$$

Using a step size h = 0.1, find the approximate solution yi of the system at t = 0.1 using:
The 2-stage, 2nd order L-Stable Rosenbrock Method.
The 2-stage, 3rd order A-Stable Rosenbrock Method.
The linear multi-step implicit Euler scheme of order.

Solving the above problems with the methods discussed in this study, we obtained the result as shown in Table 1.

Results and Discussion

In this study, we constructed the general Rosenbrock methods of orders two and three respectively, using Taylors expansion. The stability function for each method was derived from the stability analysis or application of each stability function; criteria were set for A or L-stability. Some of these criteria involving the parameters of the methods were established.

The following major results and theorems were obtained:

- The R-stage-first order RO Method is the Euler scheme.
- Theorem: A 2-2 RO is A-Stable if

  - $\alpha_1 > 0$, $\alpha_2 > 0$ and
  - $A_1^2+A_2^2-2A_2 = B_1^2+B_2^2-2B_2$

- Theorem: A condition for L-Stability of the 2-2 RO-Method is that

$$b_{21} = \alpha_1 - \frac{\alpha_2}{C_2}(\alpha_1+C_1) \qquad \text{Provided } C_2 \neq 0$$

Conclusions

From the conditions imposed on the parameters of the 2-2 and 2-3 RO-Methods for A-or L-Stability, the need for this study has been substantially realized.

Armed with these criteria, we will be able to know in advance whether a set of parameters selected for the said methods will be A-or L-stable. This is a fair departure from the tradition of first determining the parameters and, then testing for A-or L-stability.

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