



Trends in
**Applied Sciences
Research**

ISSN 1819-3579



Academic
Journals Inc.

www.academicjournals.com

Approach to the Construction of Variable Stepsize for a Class of General Linear Methods

F.E. Bazuaye and J. Ataha
 Department of Mathematics and Computer Science, Benson Idahosa University,
 Benin City, Edo State, Nigeria

Abstract: We present a new approach to the construction of variable stepsize for a class of general linear methods for the numerical solution of ordinary differential equations. These methods provide an alternative to the Nordsieck technique of changing the stepsize of integration. Order conditions are derived using a recent approach by Albrecht and examples of methods are given which are appropriate for stiff or nonstiff systems in sequential or parallel computing environments. Numerical experiments are presented which indicate that the implementation based on variable stepsize formulation is more accurate and more efficient than the implementation based on Nordsieck's technique for second-order DIMSIMs of type 1.

Key words: Variable stepsize, general linear methods, Nordsieck techniques, discretization, stability, multistage

Introduction

Butcher (1993) introduced diagonally implicit multistage integration methods (DIMSIMs) for the numerical solution of systems of ordinary differential equations (ODEs)

$$\begin{aligned} y'(x) &= f(y(x)), x \in [x_0, X], \\ y(x_0) &= y_0, \end{aligned} \quad (1)$$

where the function $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is assumed to be sufficiently smooth. These methods are given by

$$\begin{bmatrix} Y \\ \frac{Y}{y^{[n+1]}} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hf(Y) \\ \frac{hf(Y)}{Y^{[n]}} \end{bmatrix},$$

$n = 0, 1, \dots, N-1$, $N_n = X - x_0$. Here $Y = [Y_1^T, \dots, Y_s^T]^T$ are stage values which are approximations to the solution y of (1.1) at the points $x_n + h c_i$, $i = 1, 2, \dots, s$ and $y^{[n]} = [y_1^{[n]T}, \dots, y_r^{[n]T}]^T$ are approximations at the integration step n . Assuming that

$$y_i^{[n]} = \sum_{k=0}^p \alpha_{ik} h^k y^{(k)}(x_n) + O(h^{p+1})$$

for some scalars α_{ik} and that

$$Y(x_n + c_i h) = y(x_n + c_i h) + O(h^{p+1}),$$

conditions are derived by Butcher (1993) which guarantee that

Corresponding Author: Bazuaye, F.E., Department of Mathematics and Computer Science, Benson Idahosa University, Benin City, Edo State, Nigeria

$$y_i^{[n+1]} = \sum_{k=0}^p \alpha_{ik} h^k y^{(k)}(x_{n+1}) + O(h^{p+1}) \quad (2)$$

for some values of α_{ik} . This means that Albrecht (1985, 1987, 1989) has order p equal to the stage order q . Butcher and Jackiewicz (1993) used a similar approach to derive conditions on the coefficient matrices A , U , B and V and on the parameters α_{ik} under which Albrecht (1985, 1987, 1989) method has order p and stage order $q = p-1$.

To implement Albrecht methods in variable step mode it was proposed by Butcher (1993) to use the Nordsieck technique for changing stepsize. In the context of DIMSIMs this technique is based on the derivation of formulas of the form

$$\tilde{y}^{[n]} = h\tilde{B}(Y) + \tilde{V}y^{[n]},$$

where the components of $\tilde{y}^{[n]}$ are approximations of order p to the components of the vector

$$[y(x_n)^T, hy'(x_n)^T, \dots, h^p y^{(p)}(x_n)^T]^T,$$

and where \tilde{B} and \tilde{V} are constant matrices of appropriate dimensions. Examples of such approximations are given by Butcher (1993). Usually these matrices \tilde{B} and \tilde{V} depend on some number of free parameters which should be chosen to satisfy some compatibility conditions and to guarantee that the method corresponding to the new stepsize has good stability properties.

In this paper we propose the alternative technique for changing the stepsize of general linear methods. This approach is based on the derivation of formulas based directly on nonuniform meshes. This leads to variable stepsize formulation of DIMSIMs in which the coefficient matrices A , U , B , V , the vector c as well as parameters α_{ik} depend, in general, on the current stepsize and the past stepsize.

Let there be given a nonuniform mesh

$$x_p < x_{p+1} < \dots < x_1 < x_1 < x_2 < \dots < x_N, \quad x_N = X,$$

and let $h_n = x_{n+1} - x_n$, $n = -p, -p+1, \dots, N+1$. Put $\sigma_{n,i} = h_n - 1/h_i$, $i = 1, 2, p$ and $\sigma_n = [\sigma_{n,1} \dots \sigma_{n,p}]^T$, $n = -p, -p+1, \dots, N+1$. The gridpoints x_p, \dots, x_1 to the left of x_0 are introduced for notational convenience to simplify the formulas in the next section.

Alternatively, we would consider only the grid $(x_n)_{n=0}^N = 0$ and start the integration process at x_p , for some integer p .

We will investigate methods of the form

$$\begin{cases} Y^{[n+1]} = h_n A(\sigma_n) f(Y^{[n+1]}) + U(\sigma_n) y^{[n]}, \\ Y^{[n+1]} = h_n B(\sigma_n) f(Y^{[n+1]}) + V(\sigma_n) y^{[n]}, \end{cases} \quad (3)$$

$n = 0, 1, \dots, N-1$, where $Y_i^{[n+1]}$, $i = 1, 2, \dots, s$, are approximations (possibly of low order) to $y(x_i + c_i(\alpha_n)h_n)$ and the starting values $y_i^{[0]}$, $i = 1, 2, \dots, r$, are approximations to the combination of $y(x_p)$, $y(x_{p+1}), \dots, y(x_0)$. This will be made more precise in the next section. The coefficient matrices $A(\alpha_n)$, $U(\alpha_n)$, $B(\alpha_n)$ and $V(\alpha_n)$ have dimensions $s \times s$, $s \times r$, $r \times s$ and $r \times r$, respectively, where s stand for the number of internal stages and r for the number of external stages.

The product of matrices $V(\alpha_n)$ determines stability properties of (1.3). The Albrecht method is said to be zero-stable if the product $\prod_{j=0}^n V(\alpha_{n-j})$ is bounded uniformly with respect to n . We will

usually assume in this study that $V(\alpha)$ is a rank-one matrix, i.e., $V(\alpha) = \mathbf{e}\mathbf{v}^T(\alpha_n)$, where $\mathbf{e} = [1, \dots, 1]^T \in \mathbb{R}^r$ and $\mathbf{v}(\alpha_n) = [v_1(\alpha_n), \dots, v_r(\alpha_n)]^T$ and that $\mathbf{v}(\alpha_n)\mathbf{e} = 1$. These conditions guarantee that Albrecht methods is zero-stable since in this case the product

$$\prod_{j=0}^n V(\sigma_{n-j}) = \prod_{j=0}^n \mathbf{e}\mathbf{v}^T(\sigma_{n-j}) = \mathbf{e}\mathbf{v}(\sigma_0)$$

is also a rank-one matrix with a single nonzero eigenvalue equal to one.

Order Conditions

To simplify the presentation we will derive the order conditions in the scalar case $m = 1$. As in the results can be easily generalized to the vector case $m > 1$ by using the tensor product notation.

It will always be assumed that the vector $\mathbf{Y}^{[n]}$ of stage values is an approximation of at least order one to the vector function $\mathbf{z}_1(x_n)$ denoted by

$$\mathbf{z}_1(x_n) := \mathbf{y}(x_n + (\mathbf{c}(\sigma_{n-1}) - \mathbf{e}) h_{n-1}) \quad (4)$$

with the notation

$$\mathbf{y}(x_n + \mathbf{a}h_{n-1}) = [\mathbf{y}(x_n + \mathbf{a}_1 h_{n-1}), \dots, \mathbf{y}(x_n + \mathbf{a}_s h_{n-1})]^T,$$

$n = 0, 1, \dots, N$, where \mathbf{y} is the solution to (1). To derive order conditions for variable stepsize method (5), we assume that

$$\mathbf{y}^{[n]} = \sum_{l=0}^p \beta_l \mathbf{y}(x_{n-l}) + O(h^{p+1}), \quad (5)$$

$h = \max h_n$, for some constant vectors

$$\beta_1 = [\beta_{1,1}, \beta_{2,1}, \dots, \beta_{r,1}]^T,$$

and request that

$$\mathbf{y}^{[n+1]} = \sum_{l=0}^p \beta_l \mathbf{y}(x_{n+1-l}) + O(h^{p+1}),$$

for the same vectors β_1 . This is equivalent to saying that the correct value function (compare Hairer and Wanner (1991)) is defined by

$$\mathbf{z}_2(x_n) := \sum \beta_l \mathbf{y}(x_{n-l}).$$

This gives rise to a method of order p provided there exists a starting procedure to generate the initial vector $\mathbf{y}^{[0]}$ such that

$$\mathbf{y}^{[0]} = \sum_{l=0}^p \beta_l \mathbf{y}(x_{n-l}).$$

Following (3) define the local discretization errors $h_n d_1(x_{n+1})$ and $h_n d_2(x_{n+1})$ by

$$\begin{cases} \mathbf{z}_1(x_{n+1}) = h_n \mathbf{A}(\sigma_n) \mathbf{f}(\mathbf{z}_1(x_{n+1})) + \mathbf{U}(\sigma_n) \mathbf{z}_2(x_n) + h_n \mathbf{d}_1(x_{n+1}), \\ \mathbf{z}_2(x_{n+1}) = h_n \mathbf{B}(\sigma_n) \mathbf{f}(\mathbf{z}_1(x_{n+1})) + \mathbf{V}(\sigma_n) \mathbf{z}_2(x_n) + h_n \mathbf{d}_2(x_{n+1}). \end{cases} \quad (6)$$

We have

$$\begin{aligned} h_n d_1(x_{n+1}) &= y(x_n + c(\sigma_n)h_n) - U(\sigma_n) \sum_{l=0}^p \beta_l y(x_{n-l}) - h_n A(\sigma_n) y'(x_n + c(\sigma_n)h_n), \\ h_n d_2(x_{n+1}) &= \sum_{l=0}^p \beta_l y(x_{n+l-1}) - V(\sigma_n) \sum_{l=0}^p \beta_l y(x_{n-l}) - h_n B(\sigma_n) y'(x_n + c(\sigma_n)h_n), \end{aligned}$$

and expanding

$y(x_n + c(\sigma_n)h_n)$ and $y'(x_n + c(\sigma_n)h_n)$ about the point x_n after some computations we obtain

$$\begin{aligned} h_n d_1(x_{n+1}) &= C_0(\sigma_n) y(x_n) + \sum_{\mu=1}^p h_n^\mu C_\mu(\sigma_n) y^{(\mu)}(x_n) + O(h^{p+1}), \\ h_n d_2(x_{n+1}) &= \hat{C}_0(\sigma_n) y(x_n) + \sum_{\mu=1}^p h_n^\mu \hat{C}_\mu(\sigma_n) y^{(\mu)}(x_n) + O(h^{p+1}), \end{aligned}$$

where

$$\begin{cases} C_0(\sigma_n) := e - U(\sigma_n) \sum_{l=0}^p \beta_l, \\ C_\mu(\sigma_n) := \frac{c(\sigma_n)^\mu}{\mu!} - \frac{(-1)^\mu}{\mu!} U(\sigma_n) \sum_{l=0}^p \left(\sum_{v=1}^l \sigma_{n,v} \right)^\mu \beta_l - A(\sigma_n) \frac{c(\sigma_n)^{\mu-1}}{(\mu-1)!}, \end{cases} \quad (7)$$

$\mu = 1, 0, \dots, p$ and

$$\begin{cases} \hat{C}_0(\sigma_n) := (I - V(\sigma_n)) \sum_{l=0}^p \beta_l, \\ \hat{C}_0(\sigma_n) := \frac{\beta_0}{\mu!} + \frac{(-1)^\mu}{\mu!} \left(\sum_{l=2}^p \beta_l \left(\sum_{v=1}^{l-1} \sigma_{n,v} \right)^\mu - V(\sigma_n) \sum_{l=1}^p \beta_l \left(\sum_{v=1}^l \sigma_{n,v} \right)^\mu \right) - B(\sigma_n) \frac{c(\sigma_n)^{\mu-1}}{(\mu-1)!}, \end{cases} \quad (8)$$

$\mu = 1, 0, \dots, p$ and

Method (3) is said to be preconsistent if $C_0(\sigma_n) = 0$ and $\hat{C}_0(\sigma_n) = 0$. This method is said to be consistent if it is preconsistent and, in addition, $C_1(\sigma_n) = 0$ and $\hat{C}_1(\sigma_n) = 0$. Observe that the condition $C_1(\sigma_n) = 0$ means that method (1.3) has stage order at least one and that preconsistency implies that $\sum_{l=0}^p \beta_l = e$.

Subtracting (3) from (6) we obtain

$$q(x_{n+1}) = A(\sigma_n) q(x_n) + h_n B(\sigma_n) t(x_{n+1}) + h_n d(x_{n+1}), \quad (9)$$

$n = 0, 1, \dots, N-1$, where

$$q(x_n) = \begin{bmatrix} q_1(x_n) \\ q_2(x_n) \end{bmatrix} = \begin{bmatrix} z_1(x_n) - Y^{[n]} \\ z_2(x_n) - y^{[n]} \end{bmatrix}, \quad t(x_n) = \begin{bmatrix} t_1(x_n) \\ t_2(x_n) \end{bmatrix} = \begin{bmatrix} f(z_1(x_n)) - f(Y)^{[n]} \\ f(z_2(x_n)) - f(y)^{[n]} \end{bmatrix},$$

$$d(x_n) = \begin{bmatrix} d_1(x_n) \\ d_2(x_n) \end{bmatrix}, \quad A(\sigma_n) = \begin{bmatrix} 0 & U(\sigma_n) \\ 0 & V(\sigma_n) \end{bmatrix}, \quad B(\sigma_n) = \begin{bmatrix} U(\sigma_n) & 0 \\ V(\sigma_n) & 0 \end{bmatrix},$$

The solution to (9) is

$$q(x_n) = \prod_{v=0}^{n-1} A(\sigma_{n-v-1}) q(x_0) + \sum_{l=0}^{n-1} h_l \prod_{v=0}^{n-l-2} A(\sigma_{n-v-1}) B(\sigma_l) t(x_{l+1}) \\ + \sum_{l=0}^{n-1} h_l \prod_{v=0}^{n-l-2} A(\sigma_{n-v-1}) d(x_{l+1}), \quad (10)$$

$n = 1, 2, \dots$. It can be verified that, for $0 \leq l \leq n-2$, we have

$$\prod_{v=0}^{n-l} A_{v+1} = \left[\frac{0 | U(\sigma_{n-l}) \prod_{v=0}^{n-l-2} V(\sigma_{v+1}) B(\sigma_l) }{0 | \prod_{v=0}^{n-l-2} V(\sigma_{v+1})} \right] \\ \prod_{v=0}^{n-l} A_{v+1} B_l = \left[\frac{0 | U(\sigma_{n-l}) \prod_{v=0}^{n-l-2} V(\sigma_{v+1}) B(\sigma_l) }{0 | \prod_{v=0}^{n-l-2} V(\sigma_{v+1}) B(\sigma_l)} \right]$$

and this leads to the following expressions for $q_1(x_n)$ and $q_2(x_n)$:

$$q_1(x_n) = U(\sigma_{n-1}) \prod_{v=1}^{n-1} V(\sigma_{n-v-1}) q_2(x_0) + h_{n-1} A(\sigma_{n-1}) t_1(x_n) + h_{n-1} d_1(x_n) \\ + \sum_{l=0}^{n-2} h_l U(\sigma_{n-1}) \prod_{v=1}^{n-l-2} V(\sigma_{n-v-1}) B(\sigma_l) t_1(x_{l+1}) \\ + \sum_{l=0}^{n-2} h_l U(\sigma_{n-1}) \prod_{v=1}^{n-l-2} V(\sigma_{n-v-1}) d_2(x_{l+1}), \quad (11) \\ q_2(x_n) = \prod_{v=1}^{n-1} V(\sigma_{n-v-1}) q_2(x_0) + h_{n-1} B(\sigma_{n-1}) t_1(x_n) + h_{n-1} d_2(x_n) \\ + \sum_{l=0}^{n-2} h_l \prod_{v=0}^{n-l-2} V(\sigma_{n-v-1}) B(\sigma_l) t_1(x_{l+1}) \\ + \sum_{l=0}^{n-2} h_l \prod_{v=1}^{n-l-2} V(\sigma_{n-v-1}) d_2(x_{l+1}).$$

These expressions lead to the following general order criterion for variable stepsize methods (3).

Theorem 1

Assume that method (3) is zero-stable (i.e., the products $\prod_{v=0}^n V(\sigma_{n-v-1})$ are uniformly bounded

$q_2(x_0) = O(h^p)$, $d_2(x_{l+1}) = O(h^p)$, $l = 0, 1, \dots, n-1$ and that

$$B(\sigma_l) t_1(x_{l+1}) = O(h^p),$$

$l = 0, 1, \dots, n-1$. Then

$$q_2(x_n) = O(h^p),$$

and

$$q_1(x_n) = h_{n-1} A(\sigma_{n-1}) t_1(x_n) + h_{n-1} d_1(x_n) + O(h^p),$$

where

$$h_{n-1}d_1(x_n) = C_0(\sigma_{n-1})y(x_{n-1}) + \sum_{\mu=1}^p h_{n-1}^{\mu} C_{\mu}(\sigma_{n-1})y^{(\mu)}(x_{n-1}) + O(h^{p+1}),$$

and where the coefficients $C_{\mu}(\sigma_n), \mu = 0, 1, \dots, p$, are given by (7).

It follows from this theorem that if the zero-stable method satisfies $d_2(x_{n+1}) = O(h^p)$, $1 = 1, 2, \dots, n-1$ and (10) then it is convergent with order p and the error $q_1(x_n)$ of the vector of stage values $Y^{[n]}$ is given by (11).

Similarly as in [13], we can express $t_1(x_{n+1})$ in terms of $q_1(x_{n+1})$. Let

$$g_v(x_n + c_j(\sigma_n)h_n) = \frac{(-1)^{v+1}}{v!} D_y^v f(y(x_n + c_j(\sigma_n)h_n)).$$

Then

$$g_v(x_n + c_j(\sigma_n)h_n) = \sum_{\mu=0}^{p-1} g_v^{\mu}(x_n) \frac{(c_j(\sigma_n)h_n)^{\mu}}{\mu!} + O(h^p)$$

where $g_v^{(\mu)}$ stands for the derivative of order μ and putting

$$G_v(\sigma_n) = \text{diag}(g_v(x_n + c_1(\sigma_n)h_n), \dots, g_v(x_n + c_s(\sigma_n)h_n))$$

we obtain

$$G_v(\sigma_n) = \sum_{\mu=0}^{p-1} \frac{h_n^{\mu}}{\mu!} g_v^{(\mu)}(x_n) \Gamma_c(\sigma_n)^{\mu} + O(h^p)$$

where

$$\Gamma_c(\sigma_n) := \text{diag}(c_1(\sigma_n), \dots, c_s(\sigma_n))$$

This gives

$$t_1(x_{n+1}) = G_1(\sigma_n)q_1(x_{n+1}) + G_2(\sigma_n)(q_1(x_{n+1}) - h_n d_1(x_{n+1}))^2 + \dots + O(h^p). \quad (12)$$

Define

$$\begin{aligned} u &= q_1(x_{n+1}) - hA(\sigma_{n+1})t_1(x_{n+1}) - h_n d_1(x_{n+1}) - O(h^p), \\ v &= q_1(x_{n+1}) - G_1(\sigma_n)q_1(x_{n+1}) - G_2(\sigma_n)(q_1(x_{n+1}))^2 - \dots \end{aligned}$$

Then

$$\det \begin{bmatrix} u_t & u_q \\ v_t & v_q \end{bmatrix} \Big|_{h_n=0} = \det \begin{bmatrix} 0 & I \\ I & -G_1(\sigma_n) \end{bmatrix} = (-1)^s \neq 0$$

and it follows from the implicit function theorem that the functions $q_1(t_{n+1}) = q_1(t_{n+1}, h_n)$ and $t_1(x_{n+1}) = t_1(x_{n+1}, h_n)$ are unique in a neighborhood of $h_n = 0$ and have the following expansions

$$q_1(x_{n+1}) = \xi_2(x_n)h_n^2 + \xi_3(x_n)h_n^3 + \dots + \xi_{p-1}(x_n)h_n^{p-1} + O(h^p), \quad (13)$$

$$t_1(x_{n+1}) = \eta_2(x_n)h_n^2 + \eta_3(x_n)h_n^3 + \dots + \eta_{p-1}(x_n)h_n^{p-1} + O(h^p), \quad (14)$$

provided that method (3) is consistent. We can generate $\xi_s(x_n)$ and $\eta_s(x_n)$ recursively substituting (13) and (14) into (12). This leads to the following result.

Theorem 2

For $m = 2, 3, \dots, p-1$, the function $\xi_s(x_n)$ and $\eta_s(x_n)$ satisfy the following relations
 $\xi_s(x_n) = C_m(\sigma_n)y^{(m)}(x_n) + A(\sigma_n)\eta_{m-1}(x_n)$,
 and

$$\eta(x_n) = \sum_{l=0}^{m-2} \frac{1}{l!} \Gamma_c(\sigma_n)^l \left(g_1^l(x_n) \xi_{m-1}(x_n) + \sum_{\substack{i,j \geq 2 \\ i+j+k=m-1}} g_2^l(x_n) (\xi_i(x_n) \xi_j(x_n)) \right. \\ \left. + \sum_{\substack{i,j \geq 2 \\ i+j+k=m-1}} g_3^l(x_n) (\xi_i(x_n) \xi_j(x_n) \xi_k(x_n)) \right)$$

$n = 0, 1, \dots$, with $\eta(x_n)$

Proof

We will prove this theorem by induction with respect to m . The above relations are satisfied for $m = 2$. Assume that they hold for $m = 2$. Substituting (13), (14) and the expression for $h_n d_1(x_{n+1})$ into (11) with n replaced by $n+1$, we obtain

$$\sum_{k=2}^m \xi_k(x_n) h_n^k = \sum_{k=2}^m C_k(\sigma_n) y^{(k)}(x_n) h_n^k + A(\sigma_n) \sum_{k=2}^m \eta_k(x_n) h_n^{k+1} + O(h_n^{m+1}).$$

By comparing the coefficients of h_n^m we get

$$\xi_m(x_n) = C_m(\sigma_n) y^{(m)}(x_n) + A(\sigma_n) \eta_{m-1}(x_n),$$

which is the required relation for $\xi_m(x_n)$. Define now

$$s_m(x_n) = \sum_{k=2}^m \xi_k(x_n) h_n^k.$$

Then $q_1(x_{n+1}) = s_m(x_n) + O(h_n^{m+1})$ and substituting this relation and the expressions for $G_v(\sigma_n)$ into (14) we obtain

$$\begin{aligned} & \sum_{\mu=2}^m \eta_\mu(x_n) h_n^\mu \\ &= G_1(\sigma_n) s_m(x_n) + G_2(\sigma_n) (s_m(x_n))^2 + \dots + O(h_n^{m+1}) \\ &= \left(g_1(x_n) I + g_1'(x_n) h_n \Gamma_c(\sigma_n) + g_1''(x_n) \frac{h_n^2}{2!} \Gamma_c(\sigma_n)^2 + \dots \right) \sum_{i=2}^m h_n^i \xi_i(x_n) \\ &= \left(g_2(x_n) I + g_2'(x_n) h_n \Gamma_c(\sigma_n) + g_2''(x_n) \frac{h_n^2}{2!} \Gamma_c(\sigma_n)^2 + \dots \right) \sum_{2 \leq i+j \leq m} h_n^{i+j} \xi_i(x_n) \xi_j(x_n) \\ &= \left(g_3(x_n) I + g_3'(x_n) h_n \Gamma_c(\sigma_n) + g_3''(x_n) \frac{h_n^2}{2!} \Gamma_c(\sigma_n)^2 + \dots \right) \\ &\quad \times \sum_{2 \leq i+j+k \leq m} h_n^{i+j+k} \xi_i(x_n) \xi_j(x_n) \xi_k(x_n) + \dots + O(h_n^{m+1}). \end{aligned}$$

Comparing the coefficients of h_n^m yields the required relation for $\eta_m(x_n)$ and the proof is complete. It follows from Theorem 1 and formula (14) that method (3) has order p if

$$\hat{C}_0(\sigma_n) = 0,$$

for $\mu = 0, 1, \dots, p$ and if

$$B(\sigma_1)\eta_\mu(x_1) = 0$$

for $\mu = 2, 3, \dots, p-1$, where the functions $\eta_\mu(x_i)$ satisfy the relation given by Theorem 2. The order conditions up to $p = 4$ are listed below.

- $p = 0$ (preconsistency):
 $\hat{C}_0(\sigma_n) = 0, \quad B(\sigma_1)\eta_\mu(x_1) = 0$
- $p = 1$ (consistency):
 $\hat{C}_1(\sigma_n) = 0, \quad C_1(\sigma_n) = 0$
- $p = 2$:
 $\hat{C}_2(\sigma_n) = 0;$
- $p = 3$:
 $\hat{C}_3(\sigma_n) = 0,$
 $B(\sigma_n)C_2(\sigma_n) = 0;$
- $p = 4$:
 $\hat{C}_4(\sigma_n) = 0,$
 $B(\sigma_n)C_3(\sigma_n) = 0,$
 $B(\sigma_n)A(\sigma_n)C_2(\sigma_n) = 0,$
 $B(\sigma_n)\Gamma_c(\sigma_n)C_2(\sigma_n) = 0$

Here, $C_\mu(\sigma_n)$ and $\hat{C}_\mu(\sigma_n)$ are given by formulas (7) and (8), respectively.

Numerical experiments

Here we will try to compare the reliability and efficiency of step changing techniques based on variable stepsize formulation of DIMSIMs proposed in this study that based on the Nordsieck approach given by Butcher (1993). Consider first variable stepsize type 1 methods of order 2 and stage order 2 derived at the end which reduces to

$$\begin{bmatrix} \mathbf{A} & \mathbf{U} \\ \mathbf{B} & \mathbf{V} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \frac{5}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{array} \right]$$

When the stepsize is held constant. For comparison we also implemented the above method using Nordsieck's technique. Both methods have been applied to the problem

$$\begin{cases} y'(x) = \lambda(y - p(x)) + p'(x), & x \in [0, 20] \\ y(0) = y_0, \end{cases}$$

with exact solution

$$y(x) = (y_0 - p(0))\exp(\lambda x) + p(x),$$

on the mesh generated according to the formula

$$h_{n+1} = r^q h_n, \quad n = 0, 1, \dots,$$

where

$$q_n = \begin{cases} 1, & n = 0, 1, 4, 5, 8, 9, \dots \\ -1, & n = 2, 3, 6, 7, 10, 11, \dots \end{cases}$$

This means that the stepsize h_0 was increased twice by the factor of r , then decreased twice by the factor of r and so on. We have listed in Table 1 the error at the endpoint of the integration of the variable stepsize method versus implementation based on Nordsieck's technique for $y_0 = 2$, $\lambda = -20$, $p(x) = \sin x$, the ratios $r = 1, 5, 2, 3, 4$ and initial stepsizes

$$h_0 = \frac{1}{20}, \frac{1}{40}, \frac{1}{80} \text{ and } \frac{1}{160}.$$

Observe, for example, that for $h_0 = 1/20$ the quantity λh_0 is inside of the stability region of S for the underlying fixed-stepsize DIMSIM for $n = 4j$, $j = 0, 1, \dots$, on the boundary of S for $n = 2j+1$, $j = 0, 1, \dots$ and outside of S for $n = 4j+1$, $j = 0, 1, \dots$. For this stepsize pattern with $r = 2$ the Nordsieck technique is unstable (this is indicated by “*”) while the variable stepsize method still behaves in a very satisfactory way. As the ratio r increases, both techniques are becoming less stable but, again, variable stepsize methods outperform the

Nordsieck technique for a wider range of h_0 . Similar behavior was also observed for other examples and other step changing strategies. In Table 2 we have listed the number of floating point operations (in KFLOPS) for both implementations. It follows that for the DIMSIM type 1 of order 2 the implementation based on variable stepsize implementation is a little more than twice as efficient as the implementation based on Nordsieck's technique.

We have also tested methods of types 2, 3 and 4 derived in this study. The accuracy obtained for the type-2 method was of the same order for both implementations. Contrary to type-1 methods, the implementation based on Nordsieck's technique proved to be somewhat superior to the implementation based on variable stepsize formulation for DIMSIM's of types 3 and 4. On the other hand, the implementation based on variable stepsize formulation turned out to be more efficient than implementation based on Nordsieck's technique by factors of approximately 1.3, 1.7 and 1.1 for second-order DIMSIMs of types 2, 3 and 4, respectively.

Table 1: Error for type-1 methods, variable stepsize versus Nordsieck

h_0	$r = 1.5$		$r = 2.0$		$r = 3.0$		$r = 4.0$	
	VS	N	VS	N	VS	N	VS	N
1/20	1.78E-2	1.56E-2	3.62E-2	*	2.50E-2	*	*	*
1/40	8.39E-3	7.22E-3	1.92E-2	1.92E-2	1.12E-1	*	2.80E-2	*
1/80	9.68E-4	3.06E-4	1.43E-2	1.00E-2	2.50E-2	3.85E-2	5.15E-2	*
1/160	2.35E-3	1.98E-3	1.48E-2	3.06E-2	1.63E-2	9.60E-3	2.97E-2	3.72E-3

Table 2: KFLOPS for type-1 methods, variable stepsize versus Nordsieck

h_0	$r = 1.5$		$r = 2.0$		$r = 3.0$		$r = 4.0$	
	VS	N	VS	N	VS	N	VS	N
1/20	25	53	17	*	10	*	*	*
1/40	49	106	34	73	19	*	12	*
1/80	98	211	68	146	38	82	25	*
1/160	197	422	137	293	77	165	49	106

Conclusion

This study presents an approach to the construction of variable stepsize for a class of general linear methods for ordinary differential equations. The general form of order conditions is obtained using the recent approach by Albrecht and a recurrence relation for convenient generation of these order conditions is derived. As in the case of Diagonally Implicit Multistage Integration Methods (DIMSIMs) introduced by Butcher, these methods can be derived into four types depending on the structure of the coefficient matrix $A(\alpha_n)$. Type-1 and type-2 methods are appropriate for nonstiff or stiff systems in a sequential computing environment. Examples of methods of all four types are given for $s = r = p = 2$, where s is the number of internal stages, r is the number of external stages and p is the order of the method. These examples are constructed in such a way that they reduce to DIMSIMs with good stability properties when the stepsize is kept constant.

Variable stepsize methods provide an alternative to the Nordsieck's technique of changing the stepsize of integration. Numerical experiments presented in the previous section indicate the technique of changing stepsize based on variable coefficient formulation of these methods has better stability properties than that based on Nordsieck's approach for type-1 methods of order 2 derived. Both techniques of changing stepsize have similar stability properties for type-2 methods of order 2 obtained while the technique based on Nordsieck's approach is somewhat more stable than that based on variable coefficient formulation for methods of types 3 and 4 and of order 2 derived.

References

- Albrecht, P., 1985. Numerical treatment of O.D.E.S: The theory of A-methods. Numer. Math., 47: 59-87.
- Albrecht, P., 1987. A new theoretical approach to Runge-Kutta methods. Applied Math. Comp., 31: 391-406.
- Albrecht, P., 1989. Elements of a general theory of composite integration methods. Applied Math., 31: 1-17.
- Butcher, J.C., 1993. Diagonally implicit multi-stage integration methods. Applied Numer. Math., 11: 347-363.
- Butcher, J.C. and Z. Jackiewicz, 1993. Diagonally implicit general linear methods for ordinary differential equations. BIT., 33: 452-472.
- Hairer, E. and G. Wanner, 1991. Solving Differential Equations II: Stiff and Differential-Algebraic Problems (Springer-Verlag).