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Chi-square Mixture of Erlang Distributions

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Abstract: In this study, Chi-Square mixture of Erlang distribution has been defined and determines some characteristics of the distribution. Let X follows a chi-square distribution with v d.f. and Y follows a erlang distribution with parameters α and β then Z = f (x,y) follows chi square mixture of Erlang distribution with parameters v, α and β. The mean, variance, skewness and kurtosis of the distribution be

$$\frac{\alpha+v}{\alpha\beta}, \frac{\alpha+3v}{(\alpha\beta)^2}, \frac{(2\alpha+16v)^2}{(\alpha+3v)^3} \text{ and } 3+\frac{6\alpha+124v}{(\alpha+3v)^2},$$

respectively. This distribution is always positively skewed and leptokurtic for any value of the parameters.

Key words: Erlang distribution, chi-square mixture, mixture distribution, positively skewed and leptokurtic

Introduction

A mixture of distributions is a weighted average of probability distribution with positive weights that sum to one. The distributions thus mixed are called the components of the mixture. The weights themselves comprise a probability distribution called the mixing distribution. Because of these weights, a mixture is in particular again a probability distribution. Pearson (1984) was the first researcher in the field of mixture distributions who considered the mixture of two normal distributions. After the work of Pearson there was a log gap in the field of mixture distributions. Mendenhall and Haider (1958) studied on the estimation of parameters of mixed exponentially distributed failure time distributions form censored lifetime data. In the same year, Ashford (1958) did another type of work, which was different from the work of Mendenhall and Haider. Decay (1964) has improved the work of Karl Pearson, Hasselbled (1968) studied in grater detail about the finite mixture of distributions. For an example, the weights of male population for a particular country follow a normal distribution approximately and the weights of female population for that country follow another normal distribution approximately. Then the probability distribution of the weights of population for that country will be, to the same degree of approximation, a mixture of two, normal distribution. Two separate normal distributions are the components, and the mixing distribution is the simple one on the dichotomy male-female population, with the weights given by the relative frequencies of male and female population of that area.

The probability density function of non-central chi-square distribution with degrees of freedom p and non-centrality parameter λ is given below

$$f(x|\lambda;p)=\sum_{k=0}^{\infty}\frac{\chi^{\frac{p}{2}+k-1}e^{-\frac{x}{2}}}{2^{\frac{p}{2}+k}\Gamma(\frac{p}{2}+k)}.\frac{e^{-\lambda}\lambda^{k}}{k!} \quad (1)$$

Which is considered as a Poisson mixture of chi-square distributions.

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Definition of Different Mixture Distributions

Definition 1

A random variable X is said to have a Poisson mixture of distributions Roy *et al.* (1992) if its density function is given by

$$\sum_{\theta=0}^{\infty} \frac{e^{-\lambda}\lambda^{\theta}}{\theta!} f(x;\theta) \tag{2}$$

Definition 2

A random variable X is said to have a binomial mixture of distributions Roy *et al.* (1993) if its density function is given by

$$\sum_{\theta=0}^{N} \binom{N}{\theta} p^{\theta}(1-p)^{N-\theta} f(x;\theta) \tag{3}$$

where $f(x;\theta)$ is a density function.

Definition 3

A random variable X is said to have a negative binomial mixture of normal moment distributions Roy and Sinha (1995) if its density function is given by

$$\sum_{k=0}^{n} \binom{k+r-1}{r-1} p^r q^k \frac{e^{-\frac{x^2}{2}} x^{2r}}{2^{r+1/2}\Gamma(r+1/2)} \tag{4}$$

$$-\infty < x < \infty$$

Definition 4

Zaman *et al.* (2003) defined that a random variable X is defined to be a chi-square mixture of binomial distributions with v d.f. and parameters n and p, if its density function is defined by

$$f(x; v, n, p) = \int_0^{\infty} \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \binom{n+\chi^2}{x} p^x \; (1-p)\; n+\chi^2-x \; d\chi^2 \tag{5}$$

$$;x = 0,1,2,\ldots\ldots\ldots\ldots, n+\chi^2 \; n>0 \text{ and } 0<p<1$$

Definition of Chi-square Mixtured Distribution

Definition 1

A random variable X having the density function

$$f(x; v, n, p) = \int_0^{\infty} \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} g(x;\theta,p)\, d\chi^2; -\infty < x < \infty \tag{6}$$

is said to have a chi-square mixtured distributions with n d.f. where g (x;θ,p) is a density function. The name chi-square mixture of distributions is due to the fact that integral values f (x;v,θ,p) in the derived distribution in Eq. 6 is equal to one with weights equal to the ordinates of chi-square distribution having v d.f.

Definition 2

A random variable X is defined to have a chi-square mixture of Erlang distributions with v d.f. and parameters α and β, if its density function is defined as:

$$f(x; v, \alpha, \beta) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot \frac{(\alpha\beta)^{\alpha+\chi^2}}{\Gamma(\alpha+\chi^2)} e^{-\alpha\beta x}\, x^{\alpha+\chi^2-1}\, d\chi^2$$

$$0 < x < \infty \text{ and } \alpha\beta > 1 \tag{7}$$

Different moments, characteristic function and shape characteristics of the distribution are presented in the form of the following some theorems.

Theorem 1

If X follows a chi-square mixture of Erlang distributions with v d.f. and parameters α and β, then the r th raw moment about origin is given by

$$\mu_r^{/} = (\alpha\beta)^{-r} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot \frac{\Gamma(\alpha+\chi^2+r)}{\Gamma(\alpha+\chi^2)}\, d\chi^2$$

$$\text{and mean} = \frac{\alpha+v}{\alpha\beta} \qquad \text{Variance} = \frac{\alpha+3v}{(\alpha\beta)^2}$$

$$\beta_1 = \frac{(2\alpha+16v)^2}{(\alpha+3v)^3} \qquad \beta_2 = 3 + \frac{6\alpha+124v}{(\alpha+3v)^2}$$

Proof

The rth raw moment about origin is given by

$$\mu_r^{/} = E[x^r]$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot \frac{(\alpha\beta)^{\alpha+\chi^2}}{\Gamma(\alpha+\chi^2)} \int_0^\infty e^{-\alpha\beta x}\, x^{\alpha+\chi^2-1}\, dx\, d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot \frac{(\alpha\beta)^{\alpha+\chi^2}}{\Gamma(\alpha+\chi^2)} \cdot \frac{\Gamma(\alpha+\chi^2+r)}{\Gamma(\alpha\beta)^{\alpha+\chi^2+r}}\, d\chi^2$$

$$\mu_1^{/} = (\alpha\beta)^{-r} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot \frac{\Gamma(\alpha+\chi^2+r)}{\Gamma(\alpha+\chi^2)}\, d\chi^2 \tag{8}$$

If r = 1, then we get from the Eq. 8

$$\mu_1' = \text{mean} = (\alpha\beta)^{-1} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot \frac{\Gamma(\alpha+\chi^2+1)}{\Gamma(\alpha+\chi^2)}\, d\chi^2$$

$$= (\alpha\beta)^{-1} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot (\alpha+\chi^2)\, d\chi^2$$

$$= (\alpha\beta)^{-1} (\alpha+v)$$

$$= \frac{\alpha+v}{\alpha\beta}$$

If r = 2, then from the Eq. 8 we get

$$\mu_2' = (\alpha\beta)^{-2} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot (\alpha+\chi^2)(\alpha+\chi^2+1)\, d\chi^2$$

$$= (\alpha\beta)^{-2} \{\alpha(\alpha+1) + v(2\alpha+1) + v(v+2)\}$$

If r = 3, then from the Eq. 8 we get

$$\mu_3' = (\alpha\beta)^{-3} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot (\alpha+\chi^2)(\alpha+\chi^2+1)(\alpha+\chi^2+2)\, d\chi^2$$

$$= (\alpha\beta)^{-3} \{\alpha(\alpha+1)(\alpha+2) + v(2\alpha+1)(\alpha+2) + v\alpha(\alpha+1) + v(v+2)3(\alpha+1) + v(v+2)(v+4)\}$$

$$= (\alpha\beta)^{-3} \{\alpha(\alpha+1)(\alpha+2) + v(3\alpha^2+6\alpha+2) + 3v(v+2)(\alpha+1) + v(v+2)(v+4)\}$$

If r = 4, then from the Eq. 8 we get

$$\mu_4' = (\alpha\beta)^{-4} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\, \Gamma\frac{v}{2}} \cdot (\alpha+\chi^2)(\alpha+\chi^2+1)(\alpha+\chi^2+2)(\alpha+\chi^2+3)\, d\chi^2$$

on simplifications

$$\begin{aligned} = (\alpha\beta)^{-4} \{ & \alpha(\alpha+1)(\alpha+2)(\alpha+3) + v\alpha(\alpha+1)(\alpha+2) + v(2\alpha+1)(\alpha+2)(\alpha+3) \\ & + v\alpha(\alpha+1)(\alpha+3) + v(v+2)(2\alpha+1)(\alpha+2) + v(v+2)\alpha(\alpha+1) \\ & + 3v(v+2)(\alpha+1)(\alpha+3) + v(v+2)(v+4)(v+3) \\ & + 3v(v+2)(v+4)(\alpha+1) + v(v+2)(v+4)(v+6) \} \end{aligned}$$

$$\therefore \ \text{mean} = \mu_1' = \frac{\alpha+v}{\alpha\beta}$$

$$\text{Variance} = \mu_2' - \mu_1'^2$$

$$= (\alpha\beta)^{-2} \{\alpha(\alpha+1) + v(2\alpha+1) + v(v+2)\} - \frac{(\alpha+v)^2}{(\alpha\beta)^2}$$

$$= (\alpha\beta)^{-2} \{\alpha+v+2v\}$$

$$= (\alpha\beta)^{-2} (\alpha+3v)$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \frac{2a + 16v}{(\alpha\beta)^3}$$

and $\mu_4 = \mu_4' - 4\ \mu_3'\ \mu_3' + 6\ \mu_2'\ \mu_1^2 - 3\ \mu_1'^4$

on simplification we get

$$= 3\alpha^2 + 6\alpha + 18\alpha v + 124v + 27v^2$$

$$= 3(\alpha + 3v)^2 + 6\alpha + 124v$$

Now

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$= \frac{(2\alpha + 16v)^2}{(\alpha + 3v)^3}$$

and

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$= \frac{3(\alpha + 3v)^2 + 6\alpha + 124v}{(\alpha + 3v)^2}$$

$$= 3 + \frac{6\alpha + 124v}{(\alpha + 3v)^2}$$

Theorem 2

If X follows a chi-square mixture of Erlang distributions with v d.f. and parameters α and β, then the characteristic function is given by

$$\Phi_x(t) = (1 - \frac{it}{\alpha\beta})^{-\alpha}\ \{\ 1 + 2\log(1 - \frac{it}{\alpha\beta})\}^{-\frac{v}{2}}$$

and hence we can calculate mean, variance, β_1 and β_2 of the distribution.

Proof

The characteristic function of the random variable X is given by

$$\Phi_x(t) = E[e^{itx}]$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}\ (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\ \Gamma\frac{v}{2}} \cdot \frac{(\alpha\beta)^{\alpha+\chi^2}}{\Gamma(\alpha + \chi^2)} \int_0^\infty e^{-\alpha\beta x}\ x^{(\alpha+\chi^2-1)}\ dx\ \ d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}\ (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\ \Gamma\frac{v}{2}} \cdot \frac{(\alpha\beta)^{\alpha+\chi^2}}{\Gamma(\alpha + \chi^2)} \cdot \frac{\Gamma(\alpha + \chi^2)}{(\alpha\beta - it)^{\alpha+\chi^2}}\, d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}\ (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\ \Gamma\frac{v}{2}} (1 - \frac{it}{\alpha\beta})^{-(\alpha+\chi^2)}\ \ d\chi^2$$

$$= \frac{(1 - \frac{it}{\alpha\beta})^{-\alpha}}{2^{\frac{v}{2}}\ \Gamma(\frac{v}{2})} \int_0^\infty e^{-\frac{\chi^2}{2}}\, (\chi^2)^{\frac{v}{2}-1}\ e^{-\chi^2 \log(1-\frac{it}{\alpha\beta})}\, d\chi^2$$

$$= \frac{(1 - \frac{it}{\alpha\beta})^{-\alpha}}{2^{\frac{v}{2}}\ \Gamma(\frac{v}{2})} \int_0^\infty (\chi^2)^{\frac{v}{2}-1}\ e^{-\chi^2(\ 1/2 + \log(1-\frac{it}{\alpha\beta})}\, d\chi^2$$

$$\therefore \Phi_x(t) = (1 - \frac{it}{\alpha\beta})^{-\alpha}\ \{\ 1 + 2\log(1 - \frac{it}{\alpha\beta})\}^{\frac{v}{2}} \qquad (9)$$

Now the cumulant generating function is

$$K_x(t) = \log \Phi_x(t)$$

$$= -\alpha \log(1-\frac{it}{\alpha\beta}) - \frac{v}{2}\log\{1+2\log\ (1-\frac{it}{\alpha\beta})\}$$

$$= \alpha\{\frac{it}{\alpha\beta}+\frac{(it)^2}{2(\alpha\beta)^2}+\frac{(it)^3}{3(\alpha\beta)^3}+\frac{(it)^4}{4(\alpha\beta)^4}+------\} - \frac{v}{2}\log[1+2\{-\frac{it}{\alpha\beta}-\frac{(it)^2}{2(\alpha\beta)^2}-\frac{(it)^3}{3(\alpha\beta)^3}-\frac{(it)^4}{4(\alpha\beta)^4}-\ ------\}]$$

$$= \frac{\alpha}{\alpha\beta}.it+\frac{\alpha}{(\alpha\beta)^2}.\frac{(it)^2}{2!}+\frac{2\alpha}{(\alpha\beta)^3}.\frac{(it)^3}{3!}+\frac{6\alpha}{\alpha\beta)^4}.\frac{(it)^4}{4!}+\_--$$

$$+\frac{v}{2}.2\{\frac{it}{\alpha\beta}+\frac{(it)^2}{2(\alpha\beta)^2}+\frac{(it)^3}{3(\alpha\beta)^3}+\frac{(it)^4}{4(\alpha\beta)^4}+------\}$$

$$+\frac{v}{2}.\frac{2^2}{2}\{\frac{it}{\alpha\beta}+\frac{(it)^2}{2(\alpha\beta)^2}+\frac{(it)^3}{3(\alpha\beta)^3}+------\}^2$$

$$+\frac{v}{2}.\frac{2^3}{3}\{\frac{it}{\alpha\beta}+\frac{(it)^2}{2(\alpha\beta)^2}+------\}^3$$

$$+\frac{v}{2}.\frac{2^4}{4}\{\frac{it}{\alpha\beta}+\frac{(it)^2}{2(\alpha\beta)^2}+------\}^4+-----$$

$$= \frac{\alpha}{\alpha\beta}.it+\frac{\alpha}{(\alpha\beta)^2}.\frac{(it)^2}{2!}+\frac{2\alpha}{(\alpha\beta)^3}.\frac{(it)^3}{3!}+\frac{6\alpha}{\alpha\beta)^4}.\frac{(it)^4}{4!}\pm--$$

$$+\frac{v}{\alpha\beta}.it+\frac{3v}{(\alpha\beta)^2}.\frac{(it)^2}{2!}+\frac{16v}{(\alpha\beta)^3}.\frac{(it)^3}{3!}+\frac{124v}{\alpha\beta)^4}.\frac{(it)^4}{4!}\pm--$$

$$= \frac{\alpha+v}{\alpha\beta}.it+\frac{\alpha+3v}{(\alpha\beta)^2}.\frac{(it)^2}{2!}+\frac{2\alpha+16v}{(\alpha\beta)^3}.\frac{(it)^3}{3!}+\frac{6\alpha+124v}{\alpha\beta)^4}.\frac{(it)^4}{4!}\pm--$$

$$\therefore\ \text{mean} = k_1 = \text{Co-efficient of (it) in } k_x(t) = \frac{\alpha+v}{\alpha\beta}$$

$$\text{Variance} = k_2 = \text{Co-efficient of } \frac{(it)^2}{2!} \text{ in } k_x(t) = \frac{\alpha+3v}{(\alpha\beta)^2}$$

$$\mu_3 = K_3 = \text{Co-efficient of } \frac{(it)^3}{3!} \text{ in } k_x(t) = \frac{2\alpha+16v}{(\alpha\beta)^3}$$

$$K_4 = \text{Co-efficient of } \frac{(it)^4}{4!} \text{ in } k_x(t) = \frac{6\alpha+124v}{(\alpha\beta)^4}$$

Therefore $\mu_4 = K_4 + 3k_2^2$

$$= \frac{6\alpha+124v+3(\alpha+3v)^2}{(\alpha\beta)^4}$$

so $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$

$$= \frac{(2\alpha+16v)^2}{(\alpha+3v)^3}$$

$$= \frac{4(\alpha+8v)^2}{(\alpha+3v)^3}$$

and $\beta_2 = \frac{\mu_4}{\mu_2^2}$

$$= \frac{6\alpha+124v+3(\alpha+3v)^2}{(\alpha+3v)^2}$$

$$= 3+\frac{6\alpha+124v}{(\alpha+3v)^2}$$

Now the co-efficient of skewness is

$$\gamma_1 = \frac{\mu_3}{\sqrt{\mu_2^3}} = \frac{2\alpha + 16v}{\sqrt{(\alpha+3v)^3}} \quad (10)$$

We can say that in the Eq. 10 γ_1 always takes positive value. So the distribution is positively skewed.

Comment

The distribution is always positively skewed.
Again the co-efficient of kurtosis is

$$\gamma_2 = \beta_2 - 3 = \frac{6\alpha + 124v}{(\alpha+3v)^2} \quad (11)$$

Here for any value of α and v, γ_2 always takes positive value.

Comment: Since γ_2 is always a positive quantity. So the shape of the distribution is leptokurtic.

Some Important Properties of the Chi-square Mixture of Erlang Distributions

- Total probability is unity.

$$\text{i.e., } \int_0^\infty f(x, v, \alpha, \beta)dx = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}} \Gamma\frac{v}{2}} \cdot \frac{(\alpha\beta)^{\alpha+\chi^2}}{\Gamma(\alpha+\chi^2)} \int_0^\infty e^{-\alpha\beta x} x^{\alpha+\chi^2-1} d\chi^2 dx$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}} \Gamma\frac{v}{2}} \cdot \frac{(\alpha\beta)^{\alpha+\chi^2}}{\Gamma(\alpha+\chi^2)} \frac{\Gamma(\alpha+\chi^2)}{(\alpha\beta)^{\alpha+\chi^2}} d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}} \Gamma\frac{v}{2}} . \; d\chi^2$$

$$= 1$$

- The mean and variance of the distribution is given by $\frac{\alpha+v}{\alpha\beta}$ and $\frac{\alpha+3v}{(\alpha\beta)^2}$ respectively.

- If v = 0 thus the distribution reduces to Erlang distributions.
- If $\alpha = \beta = 1$, then the distribution reduces to the chi-square mixture of exponential distributions.
- The measures of skewness and kurtosis is given by

$$\beta_1 = \frac{(2\alpha+16v)^2}{(\alpha+3v)^3}$$

$$\beta_2 = 3 + \frac{6\alpha+124v}{(\alpha+3v)^2}$$

- The distribution is always positively skewed
- The shape of the distribution is leptokurtic.

The Estimation of the Parameters

We assume that the parameter v and α in the distribution (7) be known. Then the distribution contains only one parameter viz β.

Let us consider $X_1, X_2, X_3, \ldots\ldots\ldots\ldots X_N$ be a random sample of size n drawn from the density Then the 1st raw sample moment is obtained as

$$m_1' = \frac{1}{n}\Sigma x_i = \bar{x} \tag{12}$$

we have already find

$$\mu_1' = \frac{\alpha + v}{\alpha\beta} \tag{13}$$

By the method of moments, we equate (12) with (13) and obtain

$$\bar{x} = \frac{\alpha + v}{\alpha\hat{\beta}}$$

$$\hat{\beta} = (1 + \frac{v}{\alpha}) \cdot \frac{1}{\bar{x}}$$

The Shape Characteristics of the Distributions

A Computer programme is designed to calculate the ordinates of the distribution for different values of the parameters by Fortran (PROFOR) using numerical integration. It is observed from the Fig. 1. that the distribution is always positively skewed and the shape of the distribution is leptokurtic.

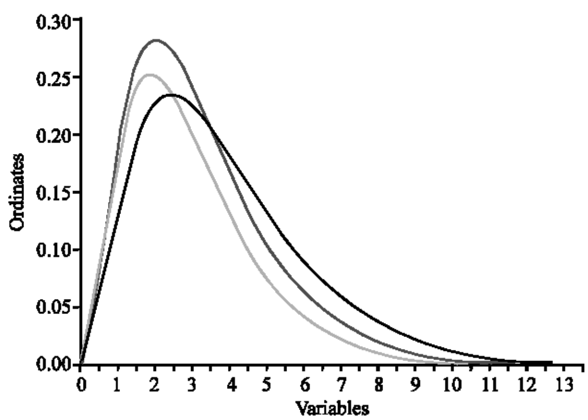

Fig. 1: The area curves of the chi-square mixture of erlang distributions

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