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On the Motion of the Frenet Vectors and Timelike Ruled Surfaces in the Minkowski 3-Space

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Abstract: In this study, we obtained the distribution parameter of a timelike ruled surface generated by a fixed straight line with respect to Frenet trihedron which is moving along a timelike curve. We show that the timelike ruled surface is constant curvature surface. Also we obtain by using director curve and base curve that necessary and sufficient conditions for the timelike ruled surface is developable in different special cases. Furthermore, we gave examples to support theorems and remarks.

Key words: Minkowski space, timelike ruled surface, Frenet vectors

INTRODUCTION

Yaylı (2002) has studied Frenet motions and spacelike ruled surfaces in the Minkowski 3-space which endoved with the standart flat metric given by $g = dx^2 + dy^2 - dz^2$ (Yaylı, 2000). In his study, he obtained the distribution parameter of a spacelike ruled surface generated by a spacelike straight line in Frenet trihedron moving along a spacelike curve and he showed that the spacelike ruled surface is developable if and only if the base curve is a helix (inclined curve).

PRELIMINARIES

The Minkowski 3-space E^3_1 is the Euclidean 3-space provided with the standart flat metric tensor given by:

$$g = -dx^2 + dy^2 + dz^2$$

where, {x, y, z} is a rectangular coordinate system of E^3_1. There are three Lorentzian causal caracters for every vectors in E^3_1. For a vector $X \in E^3_1$, it is called timelike, spacelike and null if $g(X, X)<0$, $g(X, X)>0$ and $g(X, X) = 0$, respectively. Let $\alpha: I \subset IR \rightarrow E^3_1$ be a smooth regular curve in E^3_1 where I is an open interval. The curve α is called time if $g(\alpha', \alpha')<0$ and spacelike if $g(\alpha', \alpha')>0$ and null (lightlike) if $g(\alpha', \alpha') = 0$ (Kasap, 2005).

It is well known that to each unit speed curve α with at least four countinuous derivetives, one can associate three mutually orthogonal unit vector fields T, N and B which are called the tangent, the principal normal and the binormal vector fields, respectively. At each point of the curve α, the planes Sp{T, N}, Sp{N, B} and Sp{T, B} are called, respectively as osculating, normal and rectifying planes (Ilarslan, 2005).

Let α be a timelike curve then the principal normal vector N and binormal vector field B are spacelike. We have the following Frenet formula along $\alpha(t)$:

$$\alpha'(t) = T,\ D_T T = k_1 N,\ D_T N = k_1 N + k_2 B,\ \ D_T B = -k_2 N \quad (1)$$

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where, k_1 and k_2 are curvature and torsion of α (Ikawa, 1985)

A surface in the 3-dimensional Minkowski space is called a timelike surface if the induced metric on the surface is a Lorentz metric (Beem and Ehrlich, 1981). A ruled surface is a surface swept out by a straight line X moving along a curve α. The various positions of the generating line X are called the rulings of the surface. Thus, has a parametrization in ruled form as follows;

$$\varphi\,(s,v) = \alpha\,(s)+vX(s)$$

we call α is base curve and X is the director curve. If the tangent plane is constant along a fixed ruling, than the ruled surface are called skew surfaces (Hacisalihoğlu and Turgut, 1997). If there exist a common perpendicular to two preceeding rulings in the skew surface, than the foot of the common perpendicular on the main ruling is called a cenral point. The locus of the central points is called the curve of striction.

The timelike ruled surface M in E^3_1 given by the parametrization:

$$\varphi: IxIR \rightarrow IR^3$$
$$(s,v) \rightarrow \varphi(s,v)= \alpha\,(s)+vX(s)$$

where, α is a differentiable timelike curve parametrized by its arc-lenght in E^3_1 and X(s) is the director vector of the director curve such that X is orthogonal to the tangent vector field T of the base curve α. $\{X, \bar{N}, T\}$ is an orthonormal frame field along α in E^3_1 where $\bar{N}$ is the normal vector field of M along α, X and $\bar{N}$ are spacelike and T is timelike. Thus:

$$g\,(X, X) = \varepsilon,\ g\,(\bar{N}, \bar{N}) = -g\,(T, T) = 1$$

The curve of striction of a skew timelike surface is given by:

$$\bar{\alpha}(s) = \bar{\alpha}(s) - \frac{g(T, D_T X)}{g(D_T X, D_T X)} X(s) \tag{2}$$

and $\bar{\alpha}$ is a timelike curve (Turgut, 1995). Let P_X be distribution parameter of timelike ruled surface, then:

$$P_x = -\frac{\det(T, X, D_T X)}{g(D_T X, D_T X)} \tag{3}$$

where D is Lewi-Civita connection on E^3_1 (Turgut, 1995).

Theorem 1

A timelike ruled surface is a developable surface if and only if the distribution parameter of the timelike ruled surface is zero (Turgut, 1995).

ONE-PARAMETER SPATIAL MOTION IN IR^3

Let α: $I \rightarrow E^3_1$ be a timelike curve and {T, N, B} be Frenet vectors, where T, N and B are the tangent, principal normal and binormal vectors of the curve, respectively. T is timelike and N and B are spacelike vectors.

The two coordinate systems {0; T, N, B} and $\{0';\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ are orthogonal coordinate systems in E^3_1 which represent the moving space H and the fixed space H′, respectively. Let us express the displacements of H with respect to H′ as (H/H′). In generally, during the one parameter spatial motion (H/H′), each fixed line X of the moving space H, generates a timelike ruled surface in the fixed space H′.

Let X be fixed unit spacelike vector and since $X \in Sp\{T, N, B\}$ we can write:

$$X = x_1 T + x_2 + x_3 B \tag{4}$$

such that $g(X, X) = \varepsilon$ and all x_i, $1 \le i \le 3$, are fixed and where ε is the sign of X . We can obtain the distribution parameter of the timelike ruled surface generated by line X of the moving space H. From Eq. 4 we obtain:

$$D_T X = x_1 D_T T + x_2 D_T N + x_3 D_T B \tag{5}$$

and substituting Eq. 1 into Eq. 5, we obtain,

$$D_T X = x_2 k_1 T + (x_1 k_1 - x_3 k_2) N + x_2 k_2 B$$

From Eq. 3 we obtain,

$$P_x = \frac{x_1 x_3 k_1 - (x_2^2 + x_3^2)k_2}{x_2^2(k_2^2 - k_1^2) + (x_1 k_1 - x_3 k_2)^2}, \quad -x_1^2 - x_2^2 + x_3^2 = 1 \tag{6}$$

The ruled surface developable if and only if P_X is zero (Turgut, 1995). From Eq. 6 we can state $P_X = 0$ if and only if following equation satisfies.

$$\frac{k_1}{k_2} = \frac{x_2^2 + x_3^2}{x_1 x_3}$$

We can give following remarks.

Remark 1

If k_1 and k_2 are constants (i.e., $\alpha(s)$ is helix) then P_x is constant.

Remark 2

If k_1 and k_2 are constants then timelike ruled surface $\varphi(s, v)$ is constant curvature surface.

Remark 3

Timelike ruled surface is constant curvature surface and that is:

$$K_{min} = \left(\frac{1}{P_x}\right)^2$$

Theorem 2

During the one-parameter spatial motion H/H', the timelike ruled surface in the fixed space H' generated by a fixed line X of the moving space H is developable if $\alpha(s)$ is a helix such that the harmonic curvature h of the base curve $\alpha(s)$ satisfies the equality:

$$h(s) = \frac{k_1}{k_2} = \frac{x_2^2 + x_3^2}{x_1 x_3} = \text{cons.}$$

Special Cases

The Case X is One of the Vector Fields T, N, B

In the case $X = T$ then, $x_2 = x_3 = 0$, $x_1 = 1$ and both X and T is timelike. We can easly see that $P_T = 0$ by using the Eq. 6. Thus we can gives the following remark.

Remark 4

Timelike ruled surface M is developable on the direction tangent vector field of the base curve. We can give following example for this case.

Example 1

Let us consider the curve $\alpha(s) = \left(2\sinh\left(\frac{s}{\sqrt{3}}\right), 2\cosh\left(\frac{s}{\sqrt{3}}\right), \frac{s}{\sqrt{3}}\right)$ which is timelike helix. Frenet vector fields are:

$$T = \left(\frac{2}{\sqrt{3}}\cosh\left(\frac{s}{\sqrt{3}}\right), \frac{2}{\sqrt{3}}\sinh\left(\frac{s}{\sqrt{3}}\right), \frac{1}{\sqrt{3}}\right)$$

$$N = \left(\sinh\left(\frac{s}{\sqrt{3}}\right), \cosh\left(\frac{s}{\sqrt{3}}\right), 0\right)$$

$$B = \frac{1}{\sqrt{3}}\left(\cosh\left(\frac{s}{\sqrt{3}}\right), \sinh\left(\frac{s}{\sqrt{3}}\right), 2\right)$$

and curvature and torsion are $k_1 = \frac{2}{3}$ ve $k_2 = -\frac{1}{3}$, respectively. If we take:

$$X = \left(\frac{2}{\sqrt{3}}\cosh\left(\frac{s}{\sqrt{3}}\right), \frac{2}{\sqrt{3}}\sinh\left(\frac{s}{\sqrt{3}}\right), \frac{1}{\sqrt{3}}\right)$$

then developable timelike ruled surface is

$$\varphi(t,v) = \left(\left\{\frac{2v}{\sqrt{3}}\cosh\left(\frac{s}{\sqrt{3}}\right) + 2\sinh\left(\frac{s}{\sqrt{3}}\right)\right\}, \left\{\frac{2v}{\sqrt{3}}\sinh\left(\frac{s}{\sqrt{3}}\right) + 2\cosh\left(\frac{s}{\sqrt{3}}\right)\right\}, \frac{s+v}{\sqrt{3}}\right)$$

In the case $X = N$ then $x_1 = x_3 = 0$, $x_2 = 1$ and X is space like. Thus from Eq. 6 we obtain

$$P_N = \frac{-k_2}{k_2^2 - k_1^2} \tag{7}$$

Thus we can give the following remark.

Remark 5

Timelike ruled surface is developable on the direction principal normal vector field of the base curve if and only if the base curve is planar curve with non-zero curvature.

We can give following example for this case.

Example 2

Let us consider the planar curve $\alpha(s) = (\sinh (s), \cosh (s), 0)$ which is timelike circle. Tangent and principal normal vector fields are:

$$T = (\cosh (s), \sinh (s), \cosh (s), 0)$$

and curvature and torsion are $k_1 = 1$ ve $k = 0$, respectively. If we take :

$$X = (\sinh (s), \cosh (s), 0)$$

then timelike ruled surface which developable on the principal normal vector field is

$$\varphi (s, v) = ((1+v) \sinh (s), (1+v) \cosh (s), 0)$$

In the case $X = B$ then $x_1 = x_2 = 0$, $x_3 = 1$ and X is space like. Thus from Eq. 6 we obtain

$$P_B = -\frac{1}{k_2} \tag{8}$$

Thus we can give the following remark and theorem.

Remark 6

There is no timelike ruled surface in E^3_1 which is developable on binormal vector of the base curve.

By using the Eq. 7 and 8 we can give the relation between P_N and P_B as follows.

$$\left(\frac{k_1}{k_2}\right)^2 = 1 - \frac{P_B}{P_N} \tag{9}$$

Theorem 3

During the one-parameter spatial motion H/H', the base curve α is helix (inclined curve) if and only if P_B/P_N is constant, where P_N and P_B are distribution parameters of the surfaces generated by the principal normal and binormal.

The Case X Is Lying on the Rectify, Normal or Osculator Planes of the Base Curve

In the case X is lying in the normal plane then x_1 is zero and X will be as follows:

$$X = x_2 N + x_3 B, \quad x_2^2 + x_3^2 = \varepsilon$$

We wish the timelike surface is developable surface then since from Eq. 6 we obtain

$$-(x_2^2 + x_3^2) k_2 = 0 \tag{10}$$

In this case X, N and B are spacelike and $x_2^2 + x_3^2 > 0$. Hence, from Eq. 10 we obtain $k_2 = 0$. Thus $\alpha(s)$ is a planar curve. Hence we can give the following remark and theorem.

Remark 7

Timelike ruled surfaces are developable if and only if its director curve is spacelike and it is lying normal plane of the planar base curve

Theorem 4

During the one-parameter spatial motion H/H', the timelike ruled surface in the fixed space H' generated by X in the normal plane of the base curve $\alpha(s)$ in H is developable if and only if $\alpha(s)$ is a curve in its osculating plane.

We can give following example for this case:

Example 3

Let us consider the planar timelike curve given in example 2 by taking binormal vector field as $B = e_3 = (0,0,1)$. If we take:

$$X = \frac{1}{\sqrt{2}}\left(\sinh(s), \cosh(s), 1\right)$$

then timelike ruled surface which developable on normal plane of $\alpha(s)$ is

$$\varphi(s,v) = \left(\frac{(1+v)}{\sqrt{2}}\sinh(s)\,, \quad \frac{(1+v)}{\sqrt{2}}\cosh(s)\,, \quad \frac{v}{\sqrt{2}}\right)$$

In the case X is in the osculating plane then x_3 is zero and X will be as follows:

$$X = x_1 T + x_2 N, \quad -x_1^2 + x_2^2 = \varepsilon$$

We wish the timelike surface is developable surface then since from Eq. 6 we obtain:

$$-k_2 x_2^2 = 0, \quad x_1 \neq 0 \text{ and } x_2 \neq 0 \tag{11}$$

Thus $k_2 = 0$. Hence α (s) is a planar curve. Therefore we can give following remark.

Remark 8

Timelike ruled surfaces which has timelike, spacelike and null rulings are developable if and only if its base curve is planar and its director curve is lying osculator plane of the base curve.

We can give following example for this case:

Example 4

Let us consider the planar timelike curve given in example 2. If we take:

$$X = \frac{1}{\sqrt{2}}\left(\sinh(s) + \cosh(s),\ \sinh(s) + \cosh(s), 0\right)$$

then timelike ruled surface which developable on osculator plane of α (s) is:

$$\varphi(s,v) = \left(\left(1 + \frac{v}{\sqrt{2}}\right)\sinh(s) + v\cosh(s)\,, \left(1 + \frac{v}{\sqrt{2}}\right)\cosh(s) + v\sinh(s)\ \ ,0\right)$$

In the case X is in the rectifying plane then x_2 is zero and X will be as follows:

$$X = x_1T + x_3B, \quad -x_1^2 + x_3^2 = \varepsilon$$

We wish the timelike surface is developable surface then since from Eq. 6 we obtain

$$P_x = \frac{x_3}{x_1k_1 - x_3k_2} \tag{12}$$

Thus $P_x = 0$ if and only if x_3 is zero. Therefore we can give following remark.

Remark 9

There is no developable timelike surface in E^3_1 which director curve lying in rectifiying plane of the base curve as $X = x_1\ T + x_3\ B$ where $x_1 \neq 0$ and $x_3 \neq 0$. If $x_3 = 0$ then X is the same direction of tangent vector field of α (s) and the ruled surface M is developable.

In the case the base curve α (s) is the striction curve, then $\alpha\ (s) = \bar{\alpha}(s)$. In this case we can write $g(T, D_TX) = 0$ by using Eq. 2 and we obtain $x_2\ k_1 = 0$. If $x_2 = 0$ then X is in the rectifying plane that we examined this case before. If $x_2\ k_1 \neq 0$ then the base curve is a line and X is constant with respect to both Sp{T, N, B} and $Sp\{e_1,e_2,e_3\}$. Therefore timelike ruled surface $\varphi(s, v)$ is developable.

Theorem 5

If the base curve $\alpha(s)$ is the striction curve $\bar{\alpha}(s)$ then the director curves of the timelike ruled surface, lies in the rectifying plane of both $\bar{\alpha}(s)$ and $\alpha(s)$.

Remark 10

If $\alpha(s) = \bar{\alpha}(s)$ then timelike ruled surface M is developable surface.

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