



Trends in  
**Applied Sciences  
Research**

ISSN 1819-3579



Academic  
Journals Inc.

[www.academicjournals.com](http://www.academicjournals.com)

## On the Solution of the Problem of Scheduling Unrelated Parallel Machines with Machine Eligibility Restrictions under Fuzziness

Omar M. Saad

Department of Mathematics, Faculty of Science, Helwan University,  
Ain Helwan, Postal Code 11 795, Cairo, Egypt

---

**Abstract:** This study presents a solution algorithm for the problem of minimizing the makespan on unrelated parallel machines with machine eligibility restrictions under fuzziness. It is considered that the processing times are those fuzzy parameters and the maximum completion time is required to be minimized. A simulation experiment is conducted to examine how well the fuzzy approach performs in obtaining optimal solutions with respect to the solution time. Some numerical examples are involved to clarify the developed theory and the solution algorithm.

**Key words:** Scheduling, unrelated parallel machines, makespan, fuzzy numbers,  $\alpha$ -level set

---

### INTRODUCTION

In the general case of unrelated machines, the processing time for a job on a machine is independent of its processing time on other machines. When machine eligibility restrictions are considered, certain jobs are eligible to be processed on certain machines only. The problem with unrelated parallel machines can have many applications. In a manufacturing plant, for example, many machines may be able to process a job but with different speeds depending on the machines technology. Additionally, machine eligibility restrictions can exist in airport scheduling, where planes need to be scheduled to gates. Planes correspond to jobs and gates to machines. Depending on the size of the plane, certain gates may not be able to accommodate large-sized planes.

In this study, for the parallel machines scheduling problem, the following characteristics are considered:

- The machines are unrelated;
- No job preemption (splitting) is allowed;
- The Machines cannot process two or more jobs simultaneously;
- All jobs become available for machine processing simultaneously at time zero;
- For each job  $j$ , there is a set of machines  $M_j$  ( $j = 1, \dots, n$ ) capable of processing that job;
- Processing times are fuzzy parameters.

There is an extensive literature on parallel machines scheduling that was reviewed by Cheng and Sin (1990). Many authors assume that machines are identical so that the processing time of a job does not depend on the machine to which it is assigned. It has been shown in Garey and Johnson (1979) that the problem of scheduling identical parallel machines to minimize makespan is NP-hard even for two machines. Clearly, the problem considered here is also NP-hard. Therefore, the existence of a polynomial time algorithm is highly unlikely. Consequently, for parallel machines scheduling problems of this type, most researchers have studied heuristic methods that provide an approximate solution.

Many authors in the literature address the problem of scheduling jobs on unrelated parallel machines with nonfuzzy parameters. A heuristic algorithm was developed by Horowitz and Sahni (1976) for minimizing the makespan on unrelated parallel machines that is similar to (Sahni, 1976) for identical machines. The earliest completion time heuristic (ECT) was presented by Ibarra and Kim (1977) who also developed another four heuristics that are based on ECT. The Earliest Completion Time heuristic (ECT) is similar to the longest processing time first (LPT) rule that is considered a good heuristic to minimize the makespan on identical parallel machines. The LPT rule orders the jobs by decreasing order of processing time and assigns them to machine those results in lowest partial makespan. The ECT heuristic, however, assigns at  $t = 0$  the  $m$  largest average processing time (over the machine) jobs to  $m$  machines. After that, whenever a machine is free, the unscheduled largest average processing time job is put on that machine. This heuristic tries to place the shortest average processing time jobs toward the end of the schedule where they can be used for balancing the loads. For more approximation algorithms for unrelated parallel machines without machine eligibility restrictions (Davis and Jaffe, 1981; De and Morton, 1980; Potts, 1985; Hariri and Potts, 1991). For exact and approximation algorithms by Ven de Velde (1993) and Martello *et al.* (1997).

When machine eligibility restrictions ( $M_i$ ) are involved, Pinedo (1995) showed that the Least Flexible Job first (LFJ) rule is optimal for ( $P_m/P_j = 1$ ,  $M_j/C_{max}$ ) when the  $M_i$  sets are nested. Centeno and Armacost (1997) considered machine eligibility restrictions when machines are identical and proposed an algorithm to minimize maximum lateness on identical parallel machines with release dates and machine eligibility restrictions for the special case where due dates are equal to release date plus a constant. Centeno (1998) introduced various heuristic algorithms to minimize makespan and minimize maximum lateness on identical parallel machines with machine eligibility restrictions.

In general, makespan minimization scheduling problems under fuzzy environment has received great attention in the literature and a number of reported studies have been done in this area. For more details, the reader is referred to Allet (2003), Celano *et al.* (2003), Chanas and Kasperski (2003 and 2004), Ishii *et al.* (1992) and Itoh and Ishii (1999).

Recently, Saad *et al.* (2002) presented a solution algorithm for minimizing the makespan on a job-shop scheduling problem involving fuzzy parameters in the constraints. In addition, the concept of the  $\alpha$ -level set together with the definition of the fuzzy number and its membership function have been introduced.

Lately, Salem and Armacost (2002) addressed the problem of minimizing the makespan on unrelated parallel machines with machine eligibility restrictions without fuzziness.

The primary purpose of the study here is to extend the research of Salem and Armacost (2002) and develop an exact algorithm for scheduling jobs on unrelated parallel machines with machine eligibility restrictions under fuzzy environment, where the objective is to minimize the maximum completion time and it is considered that the processing times those fuzzy parameters. Moreover, a simulation experiment is conducted to examine how well the fuzzy approach performs in obtaining optimal solutions with respect to the solution time. To our knowledge, this problem has not yet treated and discussed in the literature earlier and it seems that the suggested solution methods are new.

## **FUZZY CONCEPTS**

The fuzzy theory proposes a mathematical technique for dealing with imprecise concepts and problems that have many possible solutions. The concept of fuzzy mathematical programming on a general level was first proposed originally in the framework of the fuzzy decision of Zadeh and Bellman (1970).

Now, we introduced some necessary definitions and the reader is referred to Abass (2000), Dubois and Prade (1980).

**Definition 1**

Let  $X$  be a nonempty set. A fuzzy set  $F$  in  $X$  is characterized by its membership function  $\mu_F: X \rightarrow [0, 1]$ .

**Definition 2**

A real fuzzy number  $\tilde{a}$  is a fuzzy subset from the real line  $R$  with membership function  $\mu_{\tilde{a}}$  satisfies the following conditions:

- $\mu_{\tilde{a}}$  is a continuous mapping from  $R$  to the closed interval  $[0, 1]$ ,
- $\mu_{\tilde{a}}(x) = 0 \quad \forall x \in (-\infty, a_1]$ ,
- $\mu_{\tilde{a}}(x)$  is strictly increasing and continuous on  $[a_1, a_2]$ ,
- $\mu_{\tilde{a}}(x) = 1 \quad \forall x \in [a_2, a_3]$ ,
- $\mu_{\tilde{a}}(x)$  is strictly decreasing and continuous on  $[a_3, a_4]$ ,
- $\mu_{\tilde{a}}(x) = 0 \quad \forall x \in [a_4, +\infty)$ .

where  $a_1, a_2, a_3, a_4$  are real numbers and the fuzzy number  $\tilde{a}$  is denoted by  $\tilde{a} = [a_1, a_2, a_3, a_4]$ .

**Definition 3**

The  $\alpha$ -level set of the fuzzy number  $\tilde{a}$  is defined as the ordinary set  $L_\alpha(\tilde{a})$  for which the degree of the membership function exceeds the level  $\alpha \in [0, 1]$ :

$$L_\alpha(\tilde{a}) = \{a \in R \mid \mu_{\tilde{a}}(a) \geq \alpha\}$$

**PROBLEM STATEMENT**

Before we go any further, the unrelated parallel machines scheduling problem with machine eligibility restrictions and having fuzzy parameters in the constraints is formulated in what follows. The fuzzy parameters are the processing time of job  $j$  ( $j = 1, n$ ) on machine  $k$  ( $k = 1, m$ ). The problem of concern can be considered as a fuzzy zero-one integer problem and may be stated mathematically as follows:

$$\begin{aligned} & \text{Minimize } C_{\max} \\ & \text{subject to} \\ & C_{\max} - \sum_{j=1}^n \tilde{P}_{jk} X_{jk} \geq 0, \quad \forall k \in M_j, \\ & \sum_{k \in M_j} X_{jk} = 1, \quad 1 \leq j \leq n, \end{aligned} \tag{1}$$

$$X_{jk} = \begin{cases} 1, & \text{if job } j \text{ is assigned to machine } k, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\tilde{P}_{jk}$ ,  $1 \leq j \leq n$  and  $k \in M_j$  represent fuzzy processing time of job  $j$  on machine  $k$  and involved in the constraints of the problem under consideration, where its membership function is  $\mu_{\tilde{P}_{jk}}$

For a certain degree  $\alpha \in [0, 1]$  together with the concept of  $\alpha$ -level set of the fuzzy parameter  $\tilde{P}_{jk}$ , Problem (1) above can be understood as the following nonfuzzy unrelated machines scheduling problem with machine eligibility restrictions:

$$\begin{aligned} & \text{Minimize } C_{\max} \\ & \text{subject to} \end{aligned} \quad (2)$$

$$C_{\max} - \sum_{j=1}^n \tilde{P}_{jk} X_{jk} \geq 0, \forall k \in M_j,$$

$$\sum_{k \in M_j} X_{jk} = 1, 1 \leq j \leq n,$$

$$X_{jk} = \begin{cases} 1, & \text{if job } j \text{ is assigned to machine } k, \\ 0, & \text{otherwise,} \end{cases}$$

$$P_{jk} \in L_{\alpha}(\tilde{P}_{jk}), 1 \leq j \leq n, k \in M_j.$$

where  $L_{\alpha}(\tilde{P}_{jk})$  are the  $\alpha$ -level sets of the fuzzy parameters  $\tilde{P}_{jk}$ . Problem (2) can be rewritten in the following equivalent form:

$$\begin{aligned} & \text{Minimize } C_{\max} \\ & \text{subject to} \end{aligned} \quad (3)$$

$$C_{\max} - \sum_{j=1}^n \tilde{P}_{jk} X_{jk} \geq 0, \forall k \in M_j,$$

$$\sum_{k \in M_j} X_{jk} = 1, 1 \leq j \leq n,$$

$$X_{jk} = \begin{cases} 1, & \text{if job } j \text{ is assigned to machine } k, \\ 0, & \text{otherwise,} \end{cases}$$

$$\ell_{jk}^{(0)} \leq P_{jk} \leq L_{jk}^{(0)}, 1 \leq j \leq n, k \in M_j.$$

It should be noted that the constraint  $P_{jk} \in L_{\alpha}(\tilde{P}_{jk})$  in Problem (2) has been replaced by the equivalent one  $\ell_{jk}^{(0)} \leq P_{jk} \leq L_{jk}^{(0)}$  in Problem (3), where  $\ell_{jk}^{(0)}$  and  $L_{jk}^{(0)}$  are lower and upper bounds on the processing time  $P_{jk}$ , respectively.

### EXACT SOLUTION ALGORITHM

In what follows, we describe a solution algorithm for solving Problem (1) in finite steps. The proposed algorithm consists of two main phases. In Phase 1, Problem (1) can be converted into its equivalent nonfuzzy version. In Phase 2, where Problem (1) becomes in its nonfuzzy form, a heuristic rule is suggested to prefer between the different operations to be scheduled.

The exact solution algorithm can be described and summarized as follows:

#### Phase 1

**Step 0:** Set  $\alpha = \alpha^* = 0$ .

**Step 1:** Determine the points  $(a_1, a_2, a_3, a_4)$  for each fuzzy parameter in Problem (1) with the corresponding membership function of these fuzzy parameters.

**Step 2:** Convert Problem (1) to the nonfuzzy Problem (3).

**Phase 2**

**Step 3:** Solve the resulting problem using any available integer linear programming software package, (CPLEX 7.0 as a solver).

**Step 4:** Set  $\alpha = (\alpha^* + \text{step}) \in [0, 1]$  and go to step 1.

**Step 5:** Repeat the above procedure until the interval  $[0, 1]$  is fully exhausted. Then, stop.

**AN ILLUSTRATIVE EXAMPLE**

Suppose that there are four jobs to be processed on two unrelated machines. Let  $M_j$  be the set of machines that can process job  $j$  (the machine eligibility restrictions). Assume the  $M_j$  sets (for each job) are:

$$M_1 = \{1\}, \quad M_2 = \{2\}, \quad M_3, M_4 = \{1, 2\}.$$

Table 1 contains the fuzzy parameters which are characterized by the following fuzzy numbers:

Assume that the membership function corresponding to the fuzzy numbers is in the form (Abass, 2000):

$$\mu_{\tilde{a}}(a) = \begin{cases} 0, & a \leq a_1, \\ 1 - \left[ \frac{a - a_2}{a_1 - a_2} \right]^2, & a_1 \leq a \leq a_2, \\ 1, & a_2 \leq a \leq a_3, \\ 1 - \left[ \frac{a - a_3}{a_4 - a_3} \right]^2, & a_3 \leq a \leq a_4, \\ 0, & a \geq a_4, \end{cases}$$

where  $\tilde{a}$  corresponds to each  $\tilde{p}_{jk}$  in Problem (1). Let  $\alpha = 0.36$ , then we get:

$$\begin{aligned} 2.4 \leq P_{11} \leq 6.6, & \quad 0.6 \leq P_{32} \leq 6.4, & \quad 3.2 \leq P_{22} \leq 6.6, \\ 2.8 \leq P_{41} \leq 9.4, & \quad 0.6 \leq P_{31} \leq 5.8, & \quad 2.0 \leq P_{42} \leq 8.6. \end{aligned}$$

The nonfuzzy Problem (3) can be written as follows:

Minimize  $C_{\max}$   
subject to

$$C_{\max} - \sum_{j=1}^n \tilde{P}_{jk} X_{jk} \geq 0, \quad \forall k \in M_j,$$

$$\sum_{k \in M_j} X_{jk} = 1, \quad 1 \leq j \leq n,$$

Table 1: Fuzzy numbers

Parameters	$a_1$	$a_2$	$a_3$	$a_4$		$a_1$	$a_2$	$a_3$	$a_4$
$\tilde{p}_{11}$	2	4	5	7	$\tilde{p}_{32}$	0	3	4	7
$\tilde{p}_{22}$	3	4	5	7	$\tilde{p}_{41}$	1	5	7	10
$\tilde{p}_{31}$	0	3	5	6	$\tilde{p}_{42}$	1	6	7	9

$$X_{jk} = \begin{cases} 1, & \text{if job } j \text{ is assigned to machine } k, \\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{aligned} 2.4 \leq P_{11} \leq 6.6, & \quad 0.6 \leq P_{32} \leq 6.4, & \quad 3.2 \leq P_{22} \leq 6.6, \\ 2.8 \leq P_{41} \leq 9.4, & \quad 0.6 \leq P_{31} \leq 5.8, & \quad 2 \leq P_{42} \leq 8.6. \end{aligned}$$

The following results are obtained using AMPL software as a modeling language and CPLEX 6.0 as a solver:

$$(C_{\max}^*, X_{11}^*, X_{22}^*, X_{31}^*, X_{32}^*, X_{41}^*, X_{42}^*) = (5.2, 1, 1, 1, 0, 0, 1)$$

with the  $\alpha$ -optimal parameters:

$$(P_{11}^*, P_{22}^*, P_{31}^*, P_{32}^*, P_{41}^*, P_{42}^*) = (2.4, 3.2, 0.6, 2.8, 2).$$

## A SIMULATION EXPERIMENT

A simulation experiment is designed to assess the practicality and usefulness of the new approach in this expanded context for unrelated machine scheduling with machine eligibility restrictions under fuzzy environment. The purpose of this evaluation is to determine how well the heuristic performs in obtaining optimal solutions with respect to solution time per job.

The exact solution algorithm described before is tested against small sized problems. Only the set of problems that can be solved optimally in a reasonable amount of time is considered. The test problems range in size from 15 jobs and 2 machines to 45 jobs and 4 machines. Optimal solutions are obtained by using AMPL software to formulate the problem and CPLEX 7.0 as a solver.

The Mixed Integer Programming (MIP) formulation to Problem (3) for minimizing the makespan on unrelated parallel machines with machine eligibility restrictions using AMPL software is illustrated in Table 2.

To account for machine eligibility restrictions, jobs were assigned equally to each possible grouping of machines. For the 2 machine problem, possible groupings are machine 1, machine 2 and machines 1 and 2. With 15 jobs, five would be randomly assigned to each group and the machine eligibility sets created. For the 4 machine problem, there are 15 groups: four singletons, six pairs, four triples and one group with all four machines. When there are 15 jobs, one is assigned to each group, when there are 30 jobs, 2 are assigned to each group and so on.

The value of  $\alpha$  has been set to be 0.36. The fuzzy numbers  $a_1, a_2, a_3$  and  $a_4$  have been given values in the same manner as in the illustrative numerical example. Ten replications have been considered for each combination of machine number and job number.

Figure 1 shows the relationship between the number of jobs and the average solution time per job for  $m = 2$  and  $m = 4$ .

Table 2: MIP formulation using AMPL software

1	Set Machines;
2	Set Jobs;
3	Param Process {Jobs, Machines};
4	Var Cmax;
5	Var $\times$ {Jobs, Machines} binary;
6	Minimize Makespan:Cmax;
7	Subject to first constraint {k in Machines}:
	Cmax - sum {j in Jobs} Process [j,k] * $\times$ [j,k] $\geq$ 0;
8	Subject to second constraint {j in Jobs}: sum {k in Machines} $\times$ [j,k] = 1;

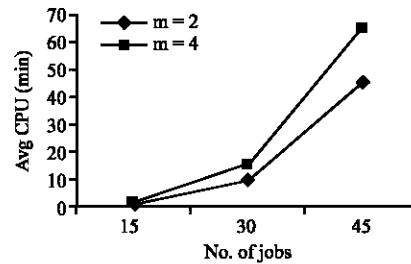


Fig. 1: No. of jobs vs. Solution time per job

## CONCLUSIONS

In this study we suggested an exact solution algorithm, described in finite steps, for solving the problem of minimizing makespan on unrelated parallel machines with machine eligibility constraints under fuzzy environment. We have considered that the processing times are those fuzzy parameters and it is required to minimize the maximum completion time (makespan). A simulation experiment has been conducted to examine how well the algorithm performs in obtaining optimal solutions with respect to solution time. Illustrated examples have been given to clarify the theory and the solution algorithm.

In fact, we conclude that it will be more interesting if the argument on what is the optimal or reasonable  $\alpha$ -level and how to evaluate the performance under the optimal or reasonable  $\alpha$ -level. We are more interested in discussing the time complexity and the worst case of algorithms. Typically, these kinds of analysis will improve theoretical values and reveal meaningful insights of applications. We should not ignore these aspects in our consideration.

## REFERENCES

- Abass, S.A., 2000. Recent approaches for treating job-shop scheduling problems with multiple objective optimization (MOJSSP). Ph. D Thesis, University of Tanta, Egypt.
- Allet, S., 2003. Handling flexibility in a generalized job shop with a fuzzy approach. *Eur. J. Operat. Res.*, 147: 312-333.
- Celano, G., A. Costa and S. Fichera, 2003. An evolutionary algorithm for pure fuzzy flowshop scheduling problems. *Int. J. Uncertainty, Fuzziness Knowledge-Based Syst.*, 11: 655-669.
- Centeno, G. and R. Armacost, 1997. Parallel machine scheduling with release time and machine eligibility restrictions. *Comput. Ind. Eng.*, 33: 273-276.
- Centeno, G., 1998. Parallel machine scheduling with specified release dates, due dates and machine eligibility restrictions for minimizing the maximum lateness. Ph.D Thesis University of Central Florida.
- Chanas, S. and A. Kasperski, 2003. On two single machine scheduling problems with fuzzy processing times and fuzzy due dates. *Eur. J. Operat. Res.*, 147: 281-296.
- Chanas, S. and A. Kasperski, 2004. Possible and necessary optimality of solutions in the single machine scheduling problem with fuzzy parameters. *Fuzzy Sets 2-c Syst.*, 142: 359-371.
- Cheng, T. and C. Sin, 1990. A state-of-the-art review of parallel-machine scheduling research. *Eur. J. Operat. Res.*, 47: 271-292.
- Davis, E. and J.M. Jaffe, 1981. Algorithms for scheduling tasks on unrelated processors. *J. Assoc. Comput. Mach.*, 28: 721-736.
- De, P. and T. Morton, 1980. Scheduling to minimize makespan on unrelated processors. *Decision Sci.*, 11: 586-602.



- Dubois, D. and H. Prade, 1980. Fuzzy Sets and Systems: Theory and applications. Academic Press, New York.
- Garey, M.R. and D.S. Johnson, 1979. Computers and Intractability: A Guide to the Theory of NP-completeness, W.H. Freeman and Company.
- Hariri, A.M.A. and C.N. Potts, 1991. Heuristics for scheduling unrelated parallel machines. *Comput. Operat. Res.*, 18: 313-321.
- Horowitz, E. and S. Sahni, 1976. Exact and approximate algorithms for scheduling nonidentical processors. *J. Assoc. Comput. Mach.*, 23: 317-327.
- Ibarra, O. and C. Kim, 1977. Heuristic algorithms for scheduling independent tasks on nonidentical processors. *J. Assoc. Comput. Mach.*, 24: 280-289.
- Ishii, H., M. Tada and T. Masuda, 1992. Two scheduling problems with fuzzy due-dates. *Fuzzy Sets Syst.*, 46: 339-347.
- Itoh, T. and H. Ishii, 1999. Fuzzy due-date scheduling problem with fuzzy processing time. *Int. Trans. Operat. Res.*, 6: 639-647.
- Martello, S., F. Soumis and P. Toth, 1997. Exact and approximation algorithms for makespan minimization on unrelated parallel machines. *Discrete Applied Math.*, 75: 169-188.
- Pinedo, M., 1995. Scheduling: Theory Algorithm and Systems. Prentice-Hall, Englewood Cliffs, New Jersey.
- Potts, C.N., 1985. Analysis of a linear programming heuristic for scheduling unrelated parallel machines. *Discrete Applied Math.*, 10: 155-164.
- Saad, M., W. Sharif and A. Abass, 2002. On the solution of the job-shop scheduling problem under fuzzy environment. *Eng. J. Univ. Qatar*, 15: 211-222.
- Sahni, S., 1976. Algorithms for scheduling with independent tasks. *J. Assoc. Comput. Mach.*, 23: 116-127.
- Salem, A. and R. Armacost, 2002. Unrelated machines scheduling with machine eligibility restrictions. *Eng. J. Univ. Qatar*, 15: 193-210.
- Ven de Velde, S., 1993. Duality-based algorithm for scheduling unrelated parallel machines. *ORSA J. Comput.*, 5: 192-205.
- Zadeh, L. and R. Bellman, 1970. Decision making in a fuzzy environment. *Manage. Sci.*, 17: 141-164.