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Homothetic Motion of the Sphere on the Plane

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Abstract: In this study, we obtained an equation of S^2 homothetic motion on its tangent plane at the contact points, along pole curves which are trajectories of instantaneous rotation centers at the contact points and we gave some remarks for the homothetic motions will be both sliding and rolling at every moments. In addition, we establish simple relationship between curvatures of the moving and fixed pole curves.

Key words: Parameter, homothetic motion, sphere

INTRODUCTION

We know that the angular velocity vector has an important role in kinematic of two rigid bodies. Müller (1966) examined the 1-parameter singular motions and he give some characterizations for axoid surfaces. Clifford and McMahon (1961) give a treatment of rolling of one curve or surface upon another during the rigid body's motion generated by the most general 1-parameter affine transformation. Nomizu (1978) studied the 1-parameter motions of unit sphere S^2 on tangent space along the pole curves using parallel vector fields at the contact points and he gave some characterizations about the angular velocity vector of rolling without sliding. Karakaş (1982) give a homothetic motion model for the unit sphere by using the technics of Nomizu (1978) and Hacisalihoğlu (1971) give some properties of 1-parameter homothetic motions. 1-parameter homothetic motion in E^3 is defined as follows:

$$F: E^3 \rightarrow E^3$$
$$x \rightarrow y = F(x) = Bx + C$$

where $A \in SO(3)$), $C \in IR^3_1$ and h is homothetic scale. The elements of A, C and h are continuously differentiable functions of time-dependent parameter t (Hacısalihoğlu, 1971). All of the homothetic motions includes both rolling and sliding at every moments and they are regular motions (Hacısalihoğlu, 1971).

Preliminaries

In this study we define the homothetic motion of unit sphere S^2 on the tangent plane of S^2 and we shall give some results and conditions using any vector fields and Frenet frames along smooth pole curves on S^2 and on tangent plane for both rolling and sliding motion. The homothetic motion of smooth surface S^2 on its tanget plane in Euclidean space of 3-dimensions is generated by the transformation

$$F: S^2 \rightarrow \Sigma$$
$$x \rightarrow y = F(x) = Bx + C, \quad B = hA \qquad (1)$$

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where A is a proper orthogonal 3x3 matrix, X and C are 3x1 vectors. The elements of A, C and h are continuously differentiable functions of time-dependent parameter t and the elements of X are coordinates of points of the curve X(t) on unit sphere. By differentiating (1) we have

$$\dot{Y} = hA\dot{X} + (\dot{h}A + h\dot{A})X + \dot{C} \tag{2}$$

where $hA\dot{X} + (\dot{h}A + h\dot{A})X + \dot{C}$ is sliding velocity of X and where (.) indicates d/dt. We called X is a pole point if sliding velocity of X is vanish and locus of points of X called pole curve (Hacısalihoğlu, 1971). We take B as hA so equation of the moving pole curve is $X = -\dot{B}^{-1}\dot{C}$. Substitution X with $X = -\dot{B}^{-1}\dot{C}$ in (1) we obtain Fixed Pole Curve $Y = -B\dot{B}^{-1}\dot{C} + C$. Now we examine the matrix $\dot{B}^{-1}$.

$$B\dot{B}^{-1} := hA(\dot{h}^{-1}A^{-1} + h^{-1}\dot{A}^{-1}) = \underbrace{h\dot{h}^{-1}I_3}_{\phi} + \underbrace{A\dot{A}^{-1}}_{S}$$

where ϕ and S is sliding part and rolling part of (1). For $S \neq 0$, there is a uniquely determined vector w such that S(U) equal to the cross product $w \times U$ for every vector $U \in IR^3$ (Appell, 1919). The vector w is called the angular velocity at instant t and the homothetic motion F in (1) called is rolling if w lies in to tangent plane of S^2 and F is spinning if w normal to tangent plane of S^2 at the contact points of S^2 and its tangent plane at instant t (Nomizu, 1978).

Sliding and Rolling of S^2 on Σ

Let us consider the unit sphere S^2 and the tangent plane Σ of S^2 at $X_0 = (0,0,1) \in S^2$, S^2. We shall take a rectangular coordinate system in E^3 such that S^2 is given by $x_1^2 + x_2^2 = 1$ and Σ is $x_3 = 1$. Let e_1, e_2 and e_3 be the unit vectors (1,0,0), (0, 1, 0) and (0,0,1), respectively. Suppose that (X) = X(t) is a moving smooth pole curve on S^2 and (Y) = Y(t) is a fixed smooth pole curve on Σ which are starting at the point x_0 for $t = t_0$. We wish to roll and slide S^2 on Σ along these curves in such a way that, at instant t the point X(t) becomes a point of contact with Y(t) on Σ. We can define homothetic motion S^2 on Σ as

$$F: S^2 \to \Sigma$$
$$x \to y = F(x) = Bx + C,\ B = hA \tag{3}$$

since $F|S^2|$ is tangent to Σ at the contact points we have $Bx = he_3$.

Suppose that $\{b_1, b_2\}$ and $\{a_1, a_2\}$ be orthonormal systems along pol curves (X) and (Y) on S^2 and Σ, respectively. Hence $\{b_1, b_2, X\}$ and $\{a_1, a_2, e_3\}$ will be moving and fixed system for (X) and (Y), respectively. In addition, assume that b_1, a_1 and b_2, a_2 transform each other as follows.

$$b_1 = hB^{-1}a_1 \quad \text{and} \quad b_2 = hB^{-1}a_2 \tag{4}$$

Remark 1

Let (X) be a curve (with the arc length parameter t) on S^2 and $\psi|(t)$ be angle between the position vector of (X) and normal vector N of (X). In this case, $\psi|(t) + k_2(t) = 0$ is satisfies. Where $k_2(t) = 0$ is satisfies. Where K_2 (t) is torsion of (X).

Proof

Let T, N, B vector fields be Frenet vectors of (X). Since all the curves on S^2 are normal curves we have

$$X(t) = \cos\psi(t)\, N(t) +|\, \sin\psi(t)\, B(t) \tag{5}$$

with the differentation both side of (5) we obtain

$$\frac{dX}{dt} = -k_1 \cos\psi T - (\psi' + k_2)\sin\psi N + (\psi' + k_2)\cos\psi B$$

thus we obtain the equations

$$\cos\psi = -\frac{1}{k_1} - (\psi' + k_2)\sin\psi = 0 \text{ and } (\psi' + k_2)\cos\psi = 0$$

and then $\psi (t) + k_2 (t) = 0$.

We must construct the frames $\{b_1, b_2, X\}$ and $\{a_1, a_2, e_3\}$ along the pole curves (X) and (Y), respectively, to determine the orthogonal matrix A. During this operations we make use of Darboux frame along (X) and (Y) at contact points on S^2 and Σ, respectively. We can find an orthogonal matrix Q using (5) then we obtain (6)

$$\begin{bmatrix} T \\ X\Lambda T \\ X \end{bmatrix} = [Q] \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{6}$$

Since the orthonormal system of {T, N, B} rotates according to $\{e_1, e_2, e_3\}$ along the curve (X) thus we can write (7) for $P \in SO(3)$,

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = [P] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{7}$$

Tangent spaces Sp $\{b_1, b_2\}$ and Sp $\{T, X\wedge T\}$ are the same space. Let's angle between b_1 and T (b_2 and $X\wedge T$, respectively) be θ. Then we can write (8)

$$\begin{bmatrix} b_1 \\ b_2 \\ X \end{bmatrix} = [R] \begin{bmatrix} T \\ X\Lambda T \\ X \end{bmatrix} \tag{8}$$

where the matrix R is a rotation matrix with the angle θ. Using the Eq. 6-8, we obtain

$$A_1 = [P]^T [Q]^T [R]^T \tag{9}$$

The matrix A_1 transforms b_1 to e_1, b_2 to e_2 and X to e_3, respectively.

On the other hand, we denote the skew symmetric matrix as $\frac{dA_1^T}{dt} A_1$ as W_1 thus W_1 will be follows.

$$W_1 = \begin{bmatrix} 0 & \theta' + k_1 \sin\psi & k_1 \cos\theta \cos\psi \\ -(\theta' + k_1 \sin\psi) & 0 & -k_1 \sin\theta \cos\psi \\ -k_1 \cos\theta \cos\psi & k_1 \sin\theta \cos\psi & 0 \end{bmatrix} \tag{10}$$

Teorem 1

b_1 and b_2 vector fields are parallel with the connection of S^2 along normal curve (X) if and only if $\theta + k_1 \sin\psi = 0$.

Proof

Let D be Levi Civita connection and S be shape operator of S^2. From (6) and (8) we can write b_1 as follows.

$$b_1 = \cos\theta\, T + \sin\theta\, \psi\, N - \sin\theta \cos\psi\, B$$

Using Gauss equations

$$\overline{D}_T b_1 = D_T b_1 + \langle S(T), b_1 \rangle X$$

we obtain

$$\overline{D}_T b_1 = -\{\theta' + k_1 \sin\psi\}\sin\theta\, T + \{k_1 \cos\psi + 1\}\cos\psi \cos\theta\, N + \{k_1 \cos\psi + 1\}\sin\psi \cos\theta\, B$$

It is easly to see that $\overline{D}_T b_1 = 0$. $\overline{D}_T b_2 = 0$ can be easly proved by using the technics of teorem 1.

On the other hand, let us $\{\overline{T}, \overline{N}, \overline{B}\}$, $\overline{k}_1$ and $\overline{k}_2$ be orthonormal frame, curvature and tosion along the pole curve (Y), respectively. Since e_3 is normal to Σ and (Y) is a planar curve, we can take $\overline{B} = e_3$. Thus the system of $\{\overline{T}, e_3 \Lambda \overline{T}, e_3\}$ will be Darboux frame along the curve (Y). We can write (11)

$$\begin{bmatrix} \overline{T} \\ e_3 \Lambda \overline{T} \\ e_3 \end{bmatrix} = \left[\overline{Q}\right] \begin{bmatrix} \overline{T} \\ \overline{N} \\ \overline{B} \end{bmatrix} \tag{11}$$

where $\overline{Q} = I_3$. Since $\{\overline{T}, \overline{N}, e_3\}$ rotates according to the system $\{e_1, e_2, e_3\}$, for $\overline{P} \in SO(3)$ we have

$$\begin{bmatrix} \overline{T} \\ \overline{N} \\ \overline{B} \end{bmatrix} = \left[\overline{P}\right] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{12}$$

Let the angle between a_1 and a_2 and $\overline{T}$ (a_2 and $e_3 \Lambda \overline{T}$ respectively) be $\overline{\theta}$. We can write (13)

$$\begin{bmatrix} a_1 \\ a_2 \\ e_3 \end{bmatrix} = \left[\overline{R}\right] \begin{bmatrix} \overline{T} \\ e_3 \Lambda \overline{T} \\ e_3 \end{bmatrix} \tag{13}$$

where the matrix $\overline{R}$ is a rotation matrix with the angle $\overline{\theta}$. Thus we obtain

$$A_2 = \left[\overline{P}\right]^T \left[\overline{Q}\right]^T \left[\overline{R}\right]^T \tag{14}$$

by using Eq. (11), (12) and (13). The matrix A_2 transforms a_1 to e_1, a_2 to e_2 and e_3 to e_3, respectively. We denote the skew symmetric matrix $\frac{dA_2^T}{dt}A_2$ as W_2 thus W_2 will be as follow.

$$W_2 = \begin{bmatrix} 0 & \overline{\theta}' + \overline{k}_1 & 0 \\ -\left(\overline{\theta}' + \overline{k}_1\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Theorem 2

a_1 and a_2 vector fields are parallel with the connection of Σ along normal curve (Y) if and only if $\overline{\theta}' + \overline{k}_1 = 0$).

Proof

Since the shape operator of Σ is $S_\Sigma = 0$, Levi Civita connection of Σ and Riemann connection of E^3 are same connections. Using (11) and (13) we can write

$$a_1 = \cos\overline{\theta}\,\overline{T} - \sin\overline{\theta}\,\overline{N}$$

and since $D_{\overline{T}}a_1 = 0$,

$$D_{\overline{T}}a_1 = \frac{da_1}{dt} = -\left(\overline{\theta}' + \overline{k}_1\right)\sin\overline{\theta}\,\overline{T} - \left(\overline{\theta}' + \overline{k}_1\right)\cos\overline{\theta}\,\overline{N}$$

we obtain $\overline{\theta}' + \overline{k}_1 = 0$. $D_{\overline{T}}{}_{a_2} = 0$ can be easly proved by using the technics of teorem 2.

Therefore, we can find the matrix A by using (9) and (14) as $A = A_2A_1^T$ so that A transforms b_{1t} (t) $a_1(t)$, $b_2(t)$ to $a_2(t)$ and X to e_3, respectively. Since the curve (X) is a pole curve, all of the points of the (X) satisfies $\dot{B}X + \dot{C} = 0$. In this case $\dot{Y} = | B\dot{X}$. Since the curve (X) is unit speed than $\frac{\dot{Y}}{h}$ = AT. Thus the homothetic scale h is h = $\left\|\dot{Y}\right\|$

The skew symmetric matrix $S = \frac{dA}{dt}A^T$ is instantaneous rotation matrix and S represents a linear transformation as S: $T_{Y(t)}\,\Sigma \rightarrow Sp\{e_3\}$. We can find the matrix S using (10) and (15) as $S = A_2\left(-W_2 + W_1\right)A_2^T$. Cosequently the matrix S determines a vector W ϵ $Sp\{\alpha_1, \alpha_2, e_3\}$. We find the vector field W as follows.

$$W = k_1\cos\psi\sin\theta\,a_1 + k_1\cos\psi\cos\theta\,a_2 - \left(\overline{\theta}' + \overline{k}_1 - \theta' - k_1\sin\psi\right)e_3 \tag{16}$$

Remark 2

If b_1, b_2 and a_1, a_2 are parallel vector fields along (X) and (Y), respectively and h = 1 then we obtain all of the results as describe earlier in (Nomizu, 1978). In this case, W vector field will be as follows.

$$W = k_1\cos\psi\sin\left(-\int_I k_1\sin\psi\,dt\right)a_1 + k_1\cos\psi\cos\left(-\int_I k_1\sin\psi\,dt\right)a_2$$

It is obvious that W lies in the tangent space of S^2 at contact points of (X) and (Y).

Remark 3

If b_1, b_2 and a_1, a_2 are parallel vector fields along (X) and (Y), respectively, then the transformation F is a homothetic motion.

Remark 4

The sphere S^2 slides and rolls on its tangent plane along geodesics.

The vector field we have defined $b_1(t)$, $b_2(t)$) and $a_1(t)$ $a_2(t)$, for the motion F, need not to be parallel along (X) and (Y), respectively. In this case, we can give remark 5 as a main remark of this study.

Remark 5

Let $\{b_1, b_2\}$ and $\{a_1, a_2\}$ are orthonormal vector fiels systems along the curves (X) and (Y), respectively. Thus F is a homotetic motion if and only if

$$\overline{\theta}' + \overline{k}_1 - \theta' - k_1 \sin\psi = 0$$

is satisfied.

Example 1

Let X(t) = (sin t, 0, cos t), $t \in [0, \pi]$ is unit speed curve on S^2 and $Y(t) = \left(\frac{t^2}{2}, \frac{t^2}{2}, 1\right)$ is any curve on Σ. We obtain

$$T = (\cos t, 0, -\sin t), \quad N = (-\sin t, 0, -\cos t), \quad B = (0, 1, 0), \quad k_1 = 1, \quad k_2 = 0, \quad \psi = \pi$$

$$\overline{T} = \frac{1}{\sqrt{2}}(1,1,0),\ \overline{N} = \frac{1}{\sqrt{2}}(1,-1,0),\ \overline{B} = (0,0,1),\ \overline{k}_1 = 0,\ \overline{k}_2 = 0,\ \overline{\psi} = \frac{\pi}{2},\ \left\|\frac{dY}{dt}\right\| = h = t\sqrt{2}$$

for (X) and (Y) curves, respectively. Since $\frac{dY}{dt} = B\frac{dX}{dt}$, we obtain $\overline{\theta}(t) = \theta = \pi$ so the motion will be as follows.

$$Y(t) = t\sqrt{2}\begin{bmatrix} \frac{\cos t}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{\sin t}{\sqrt{2}} \\ \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\sin t}{\sqrt{2}} \\ \sin t & 0 & \cos t \end{bmatrix} X(t) + \begin{bmatrix} t^2/2 \\ t^2/2 \\ 1 - t\sqrt{2} \end{bmatrix} \quad (17)$$

And the matrix S and the vector W will be as follows.

$$S = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \text{ and } \vec{W}\Big|_p = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \in Sp\{e_1, e_2\}$$

respectively. We put the values $\psi = \pi$, $k_1 = 1$, $k_2 = 0$, $\bar{k}_1 = 0$, $\bar{k}_2 = 0$, and $\bar{\theta}(t) = \theta = \pi$ in (16) we obtain $W \in Sp\{a_1, a_2, e_3\}$ as follows

$$W = (0, 1, 0)$$

Since $W \in Sp\{a_1, a_2\} \equiv Sp\{e_1, e_2\} \equiv \Sigma$ and the condition we gave in remark 5 satisfied, then the motion (17) is a homothetic motion. In addition, moving and fixed pol curves of the motion (17) is $X(t) = (\sin t, 0, \cos t)$ and $Y(t) = \left(\frac{t^2}{2}, \frac{t^2}{2}, 1\right)$

, respectively. Thus S^2 both sliding and rolling on its tangent plane Σ along the curves (X) and (Y) according to (17) (Fig. 1).

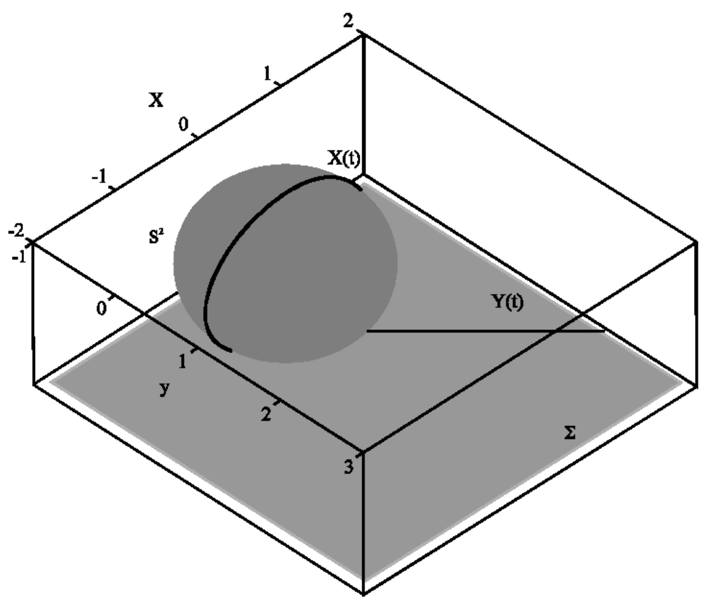

Fig. 1: Homothetic motion

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