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On Kinematics of the Hyperbolic Sphere

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Abstract: In this study, we obtained an equation of homothetic motion of hyperbolic sphere on its tangent plane along the pole curves which are trajectories of instantaneous rotation centers at the contact points and we gave some remarks for the homothetic motions will be both sliding and rolling at every moments. In addition, we establish simple relationship between curvatures of the moving and fixed pole curves.

Key words: Homothetic motion, darboux frame, darboux vector, hyperbolic sphere

INTRODUCTION

It is well known that the angular velocity vector has an important role in kinematic of two rigid bodies, especially rolling on another from (Nomizu, 1978; Appell, 1919). So mathematicians and phycisists interpreted rigid body motions in various ways. Nomizu (1978) studied the 1-parameter motions of unit sphere S^2 on its tangent space along the pole curves using parallel vector fields at the contact points and he gave some characterizations about the angular velocity vector of rolling without sliding in Euclidean case. Hacisalihoğlu (1971) has showed some properties of 1-parameter homothetic motions in Euclidean case too. Tunçer *et al*. (2007) studied Euclidean version of this study. He gave the sufficient and necessary conditions for 1-parameter homothetic motion of unit sphere S^2 on its tangent space along the pole curves.

In this study we define a homothetic motion model for the hyperbolic sphere H_0^2 on the tangent plane in Lorentzian 3-space and we shall give some results and conditions using any vector fields and Frenet frames along smooth spacelike pole curves on H_0^2 while moving on tangent plane homothetically.

PRELIMINARIES

In a Lorentzian Manifold, we can find three types of submanifolds: Space-like (or Riemannian), time-like (Lorentzian) and light-like (degenerate or null), depending on the induced metric in the tangent vector space. Lorentz surfaces has been examined in numerous articles and books. Let IR^3 be endowed with the pseudoscalar product of X and Y is defined by:

$$\langle X, Y\rangle = x_1y_1 + x_2y_2 - x_3y_3,\ X = (x_1, x_2, x_3),\ Y = (y_1, y_2, y_3)$$

$(IR^3, \langle,\rangle)$ is called 3-dimensional Lorentzian space denoted by L^3 or E_1^3 (Walrave, 1995). The Lorentzian vector product is defined by:

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$$X \wedge Y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

A vector X in L^3 is called a space-like, time-like, light-like vector if $\langle X, X\rangle > 0$, $\langle X, X\rangle < 0$ or $\langle X, X\rangle = 0$, respectively. For $X \in L^3$, the norm of X defined by $\|X\| = \sqrt{|\langle X, X\rangle|}$ and X is called a unit vector if ||X|| (Walrave, 1995).

Let $X : I \subset IR \longrightarrow E_1^3$ be a unit speed regular curve in E_1^3 where I is an open interval. The curve X(t) is called timelike if $\langle X', X'\rangle < 0$, spacelike if $\langle X', X'\rangle > 0$ and null (lightlike) if $\langle X', X'\rangle = 0$. For each curve X(t) with at least four countinuous derivetives, one can associate three mutually orthogonal unit vector fields T, N and B which are called the tangent, the principal normal and the binormal vector fields, respectively. At each point of the curve X(t), the planes Sp {T, N}, Sp {N, B} and Sp {T, B} are called, respectively as osculating, normal and rectifying planes. For the spacelike curve X(t) (with a spacelike or timelike principal normal N), the Frenet formulae read

$$T' = k_1 N,\ N' = -\varepsilon k_1 T + k_2 B,\ B' = k_2 N \tag{1}$$

where $\langle T, T\rangle = 1, \langle N, N\rangle = -\langle B, B\rangle = \epsilon$ (Ilarslan, 2005).

On the other hand, let H_0^2 be hyperbolic sphere and Σ be a plane tangent to hyperbolic sphere H_0^2 at the point (0, 0, 1) which is given by the equation $x_3 = 1$ according to Euclidean coordinate system $\{x_1, x_2, x_3\}$. One parameter homothetic motion of H_0^2 on Ó in Lorentzian space of 3-dimensions is can definite by an affine transformation:

$$\begin{aligned} F : H_0^2 &\longrightarrow \Sigma \\ X &\longrightarrow Y = hAX + C \end{aligned} \tag{2}$$

where, $A \in SO_1(3)$, X and C are 3×1 vectors. The elements of A, C and h are continuously differentiable functions of time-dependent parameter t and the elements of X are coordinates of a point in the body. It is well known that all of the homothetic motions includes both rolling and sliding at every moments and they are regular motions. By differentiating (2) we obtain:

$$Y' = hAX' + (h'A + hA')X + C' \tag{3}$$

where, $(h'A + hA')X + C'$ is sliding velocity of X. We called X is a pole point if sliding velocity of X is vanish and locus of points of X called pole curve. We take B as hA so equation of the moving pole curve is $X = -(B')^{-1}C'$. Substitution X with $X = -(B')^{-1}C'$ in (3) we obtain fixed pole curve $Y = -(B')^{-1}C' + C'$. Now we examine the matrix $B(B')^{-1}C'$.

$$B(B')^{-1} = hA\left(h'^{-1}A^{-1} + h^{-1}A'^{-1}\right) = \underbrace{hh'^{-1}I_3}_{\varphi} + \underbrace{AA'^{-1}}_{S}$$

where, φ and S is sliding part and rolling part of (2). For S≠0, there is a uniquely determined vector W(t) such that S(U) equal to the cross product W(t)×U for every vector U. The vector W(t) is called the angular velocity at instant t and the homothetic motion F in (2) called is rolling if W(t) lying in to Σ and F is spinning if W(t) normal to Σ at the contact point of H_0^2 and Σ at instant t (Nomizu, 1978).

Homothetic Motion Of H_0^2 On Σ

Let us consider the hyperbolic sphere H_0^2 and the tangent plane Σ of H_0^2 at $x_0 = (0, 0) \in H_0^2$. We shall take the equation of H_0^2 according to rectangular coordinate system in E_1^3 such as:

$$x_1^2 + x_2^2 - x_3^2 = -1$$

and Σ is $X_3 = 1$. Let e_1, e_2 and e_3 be the unit vectors (1,0,0), (0,1,0) and (0,0,1), respectively. Suppose that X(t) is a moving smooth space-like pole curve on H_0^2 starting at x_0. We wish to move H_0^2 along fixed smooth space-like pole curve on Σ homotheticaly in such a way that at instant t the point X(t) becomes a point of contact with Y(t) on Σ. From Eq. 2, since F (H_0^2) is tangent to Σ at the contact points we have: $Bx = he_3$.

Suppose that $\{b_1, b_2\}$ and $\{a_1, a_2\}$ be orthonormal systems along space-like pol curves (X) = X(t) and (Y) = Y(t) on H_0^2 and Σ, respectively. Let b_1, b_2 and a_1, a_2 be vector fields along (X) and (Y) so that:

$$b_1 = hB^{-1}a_1 \text{ and } b_2 = hB^{-1}a_2 \tag{4}$$

Hence $\{b_1, b_2, X\}$ and $\{a_1, a_2, e_3\}$ will be moving and fixed system for (X) and (Y) on H_0^2 and Σ, respectively. On the other hand, since the curve (X) is a spherical curve we can write the position vector of (X) as follows:

$$X = \lambda N + \mu B \tag{5}$$

where, the coefficients λ and μ are differentiable with respect to the parameter t.

Theorem 1

Let (X) be a space-like curve (with the pseudo arc length parameter t) on H_0^2. In this case, $1 + \epsilon \lambda k_1 = 0$, $\lambda' + \mu k_2 = 0$, $\mu' + \lambda k_2 = 0$, $\lambda\lambda' - \mu\mu' = 0$ and $\mu'\lambda - \mu'\lambda = \epsilon k_2$ are satisfies. Where k_1 and k_2 are curvature and torsion of (X), respectively.

Theorem can be easly proved by using Eq. 1 and 5 and the equalities $\langle T, T\rangle = 1$, $\langle N, N\rangle = -\langle B, B\rangle = \varepsilon$.

For determine the semi-orthogonal matrix A, we must construct the frames $\{b_1, b_2, X\}$ and $\{a_1, a_2, e_3\}$ along the pole curves (X) and (Y), respectively. During this operations we make use of Darboux frame along (X) and tangent and normal vector field of (Y) at contact points on H_0^2 and Σ, respectively. Let T, N and B vector fields be Frenet vectors of (X), so we can find semi-orthogonal matrices P, Q, R ∈ SO_1 (3) between the orthonormal systems {T, N, B} and $\{e_1, e_2, e_3\}$, {T, XΛT, X} and {T, N, B}, $\{b_1, b_2, X\}$ and {T, XΛT, X}, respectively. Hence, the matrix $A_1 = P^T, Q^T, R^T \in SO_1(3)$ transforms b_1 to e_1, b_2 to e_2 and X to e_3. Similarly, let, $\overline{T}\,\overline{N}$ and $\overline{B}$ be Frenet vectors of (Y). Since the (Y) is planar spacelike curve, binormal vector field of (Y) will be same direction with timelike vector e_3, so we take $\overline{B} = e_3$ and (Y) is spacelike curve with spacelike principal normal vector field. We can find again semi-orthogonal matrices $\overline{P}, \overline{Q}, \overline{R} \in SO_1(3)$ between the orthonormal systems $\{\overline{T}, \overline{N}, e_3\}$ and $\{e_1, e_2, e_3\}$, $\{\overline{T}, e_3\Lambda\overline{T}, e_3\}$ and $\{\overline{T}, \overline{N}, e_3\}$, $\{a_1, a_2, e_3\}$ and $\{\overline{T}, e_3\Lambda\overline{T}, e_3\}$. Thus, the matrix $A_2 = \overline{P}^T\overline{Q}^T\overline{R}^T \in SO_1(3)$ transforms a_1 to e_1, a_2 to e_2 and e_3 to e_3, respectively.

Cosequently, the matrix $A = A_2 A_1^T$ transforms b_1 to a_1, b_2 to a_2 and X to e_3.

On the other hand, we denote the skew symmetric matrix in semi-Euclidean mean $\frac{dA_1^{-1}}{dt}A_1$ and $\frac{dA_2^{-1}}{dt}A_2$ as W_1 and W_2, respectively, then W_1 and W_2 will be follows:

$$\begin{bmatrix} \frac{db_1}{dt} \\ \frac{db_2}{dt} \\ \frac{dX}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \theta' - \varepsilon\mu k_1 & -\varepsilon\lambda k_1 \cos\theta \\ -(\theta' - \varepsilon\mu k_1) & 0 & \varepsilon\lambda k_1 \sin\theta \\ -\varepsilon\lambda k_1 \cos\theta & \varepsilon\lambda k_1 \sin\theta & 0 \end{bmatrix}}_{W_1} \begin{bmatrix} b_1 \\ b_2 \\ X \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} \frac{da_1}{dt} \\ \frac{da_2}{dt} \\ \frac{de_3}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \overline{\theta}' - \overline{k}_1 & 0 \\ -\overline{\theta}' + \overline{k}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{W_2} \begin{bmatrix} a_1 \\ a_2 \\ e_3 \end{bmatrix} \tag{7}$$

where, $\theta = \theta(t)$ and $\overline{\theta} = \overline{\theta}(t)$ are angles between b_1 and T (or b_1 and $X \Lambda T$) and a_1 and $\overline{T}$ (or a_2 and $e_3 \Lambda \overline{T}$), respectively.

Remark 1

The vector fields b_1 and b_2 are parallel with the connection of H_0^2 along normal spacelike curve (X) if and only if $\theta' - \varepsilon\mu k_1 = 0$ satisfies. In this case, b_1 and b_2 has not any component in tangent space $T_{H_0^2}(X(t))$.

Remark 2

The vector fields b_1 and b_2 are parallel with the connection of Σ along spacelike curve (Y) if and only if $\overline{\theta}' - \overline{k}_1 = 0$ satisfies.

The skewsymmetric matrix (in semi-Euclidean mean) $S = \frac{dA^{-1}}{dt}A$ is instantaneous rotation matrix represents rolling part of the homothetic motion F and S represents a linear transformation defined as:

$$S: T_{Y(t)} \Sigma \rightarrow Sp\{e_3\}$$

We can find the matrix S using A_1 and A_2 as:

$$S = A_2(-W_2 + W_1)A_2^T$$

Consequently, the matrix S determines a vector:

$$W \in Sp\,\{a_1, a_2, e_3\}$$

At the contact points P = X(t), we obtain angular velocity vector W as follows:

$$W_P = \varepsilon\lambda k_1 \sin\theta(a_1)_P + \varepsilon\lambda k_1 \cos\theta(a_2)_P + \left(\overline{\theta'} - \overline{k_1} - \theta' + \varepsilon\mu k_1\right)(e_3)_P$$

Since F is a homothetic motion, H_0^2 both rolling and sliding on Σ and at the contact points W have to lie on Σ at every moments. Thus we can give following theorem:

Theorem 2

F is a rolling motion if and only if $\overline{\theta'} - \overline{k_1} - \theta' + \varepsilon\mu k_1 = 0$

Remark 3

If $\{b_1, b_2\}$ and $\{a_1, a_2\}$ are parallel vector fields along the spacelike curves (X) and (Y) then F is a rolling motion automatically. In this case W will be as follows:

$$W_P = \varepsilon\lambda k_1 \sin\left(\int_I \varepsilon k_1\mu dt\right)(a_1)_P + \varepsilon\lambda k_1 \cos\left(\int_I \varepsilon k_1\mu dt\right)(a_2)_P$$

where, I is an interval of IR which is consisting 0.

Remark 4

W will be spacelike vector during the homothetic motions.

Remark 5

Let $\{b_1, b_2\}$ and $\{a_1, a_2\}$ are any orthonormal (not parallel) vector fields systems along the spacelike curves (X) and (Y). Thus H_0^2 moves on Σ homothetically along arbitrary (X) and (Y) spacelike curves on which satisfies the conditions $\theta' - \varepsilon\mu k_1 = 0$ and $\overline{\theta'} - \overline{k_1} = 0$, respectively.

Remark 6

Let $\{b_1, b_2\}$ and $\{a_1, a_2\}$ are parallel vector fields systems along the spacelike curves (X) and (Y). Thus H_0^2 moves on Σ homothetically along spacelike geodesics.

Example

Let:

$$X(t) = \left(\frac{\sqrt{2}}{2}\sinh t, \frac{\sqrt{2}}{2}\sinh t, \cosh t\right) \quad t \in [0, \pi]$$

is unit speed curve on H_0^2 and $Y(t) = \left(\frac{t^2}{2}, \frac{t^2}{2}, 1\right)$ is any curve on Σ. We obtain:

$$T=\left(\frac{\sqrt{2}}{2}\cosh t,\frac{\sqrt{2}}{2}\cosh t,\sinh t\right),N=\left(\frac{\sqrt{2}}{2}\sinh t,\frac{\sqrt{2}}{2}\sinh t,\cosh t\right)$$

$$B=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right)$$

$$k_1=1,k_2=0,\lambda=1,\mu=0,\varepsilon=-1$$

and

$$\overline{T}=\frac{1}{\sqrt{2}}(1,1,0),\overline{N}=\frac{1}{\sqrt{2}}(1,-1,0),\overline{B}=(0,0,1),\overline{k}_1=0,\overline{k}_2=0$$

for (X) and (Y) curves. Since ||dy/dt|| = h we find $h=t\sqrt{2}$ and using $\frac{dY}{dt}=B\frac{dX}{dt}$ we obtain $\theta(t)=\overline{\theta}(t)=0$ so the motion will be as follows:

$$Y(t)=t\sqrt{2}\begin{bmatrix}\frac{1}{2}(1+\cosh t) & \frac{1}{2}(-1+\cosh t) & -\frac{\sinh t}{\sqrt{2}}\\ \frac{1}{2}(-1+\cosh t) & \frac{1}{2}(1+\cosh t) & -\frac{\sinh t}{\sqrt{2}}\\ -\frac{\sinh t}{\sqrt{2}} & -\frac{\sinh t}{\sqrt{2}} & \cosh t\end{bmatrix}X(t)+\begin{bmatrix}t^2/2\\ t^2/2\\ -t\sqrt{2}\end{bmatrix}$$

And the matrix S and the vector W will be as follows:

$$S=\begin{bmatrix}0 & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 0 & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}\text{ and } W=\left(1/\sqrt{2},-1/\sqrt{2},0\right)\text{ and } W=\left(1/\sqrt{2},-1/\sqrt{2},0\right),$$

respectively and the condition $\theta'-\varepsilon\mu k_1-\overline{\theta}'+\overline{k}_1=0$ is satisfies. Thus the motion Y = BX + C is rolling motion of H_0^2 on Σ.

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