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## Buckling Analysis of Symmetric Laminated Composite Plates by Using Discrete Singular Convolution

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**Abstract:** A numerical model for the analysis of laminated composite rectangular plates is proposed. Buckling analysis is presented for moderately thick symmetrically laminated composite plates with various boundary conditions. The formulations are based on the First-order Shear Deformation Theory (FSDT). The method of Discrete Singular Convolution (DSC) is employed for numerical solution. In the proposed approach, the derivatives in both the governing equations and the boundary conditions are discretized by the method of DSC. The results obtained by DSC method were compared with those obtained by the other numerical and analytical methods. A comparison of the results of the title problem with those of earlier studies indicates excellent agreement.

**Key words:** Discrete singular convolution, bucklin, composite plates, shannon kernel, shear deformation

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### INTRODUCTION

Laminated composite plates are widely used in the mechanical, civil, aero-space and chemical engineering. For this reason, the free vibration and buckling analyses of laminated plates have been studied by many researchers. A very detailed mathematical treatment is given in the book by Reddy (1997), Whitney (1987) and Qatu (2004). A variety of numerical and approximate methods are available today for vibration and buckling analysis of laminated plates. Among the approximate and numerical approaches used for laminated plates are the Ritz, Galerkin and Levy methods, the finite elements, differential quadrature and finite strip methods. The primary objective of this study is to give a numerical solution of buckling analysis of symmetrically laminated composite plates by the method of DSC. To the authors' knowledge, it is the first time the DSC method has been successfully applied to symmetrically laminate composite plate problems for the analysis of buckling. The procedure is based on the application of the discrete singular convolution method in conjunction with the First-order Shear Deformation Theory (FSDT).

### FUNDAMENTAL EQUATIONS

Based on the first-order shear deformation theory, the governing equations for symmetric laminates under transverse loads are given (Reddy, 1997).

$$\begin{aligned}
 & D_{11} \frac{\partial^2 \phi_x}{\partial x^2} + D_{66} \frac{\partial^2 \phi_x}{\partial y^2} + D_{16} \frac{\partial^2 \phi_y}{\partial x^2} + D_{26} \frac{\partial^2 \phi_y}{\partial y^2} + 2D_{16} \frac{\partial^2 \phi_x}{\partial x \partial y} \\
 & (D_{12} + D_{66}) \frac{\partial^2 \phi_y}{\partial x \partial y} - kA_{45} \left( \phi_y + \frac{\partial w}{\partial y} \right) - kA_{55} \left( \phi_x + \frac{\partial w}{\partial x} \right) = 0,
 \end{aligned} \tag{1a}$$

$$D_{16} \frac{\partial^2 \phi_x}{\partial x^2} + D_{26} \frac{\partial^2 \phi_x}{\partial y^2} + D_{66} \frac{\partial^2 \phi_y}{\partial x^2} + D_{22} \frac{\partial^2 \phi_y}{\partial y^2} + 2D_{26} \frac{\partial^2 \phi_y}{\partial x \partial y} \\ (D_{12} + D_{66}) \frac{\partial^2 \phi_x}{\partial x \partial y} - kA_{44} \left( \phi_y + \frac{\partial w}{\partial y} \right) - kA_{55} \left( \phi_x + \frac{\partial w}{\partial x} \right) = 0, \quad (1b)$$

$$\frac{\partial}{\partial x} \left[ kA_{45} \left( \phi_y + \frac{\partial w}{\partial y} \right) + kA_{55} \left( \phi_x + \frac{\partial w}{\partial x} \right) \right] \\ + \frac{\partial}{\partial y} \left[ kA_{44} \left( \phi_y + \frac{\partial w}{\partial y} \right) + kA_{55} \left( \phi_x + \frac{\partial w}{\partial x} \right) \right] + q(x,y) \\ + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0. \quad (1c)$$

Where  $N_x$ ,  $N_{xy}$  and  $N_y$  are the in-plane applied forces. Also, mass inertias are given as (Liew *et al.*, 2004).

$$I_0 = \int_{-h/2}^{h/2} \rho dz, \quad I_2 = \int_{-h/2}^{h/2} \rho z^2 dz \quad (2,3)$$

Where  $\rho$  and  $h$  denote the density and total thickness of the plate, respectively. The bending moments and shear forces are given as (Liew and Huang, 2003)

$$M_x = D_{11} \frac{\partial \phi_x}{\partial x} + D_{12} \frac{\partial \phi_y}{\partial y} + D_{16} \frac{\partial \phi_y}{\partial x} + D_{16} \frac{\partial \phi_x}{\partial y}, \quad (4a)$$

$$M_y = D_{12} \frac{\partial \phi_x}{\partial x} + D_{22} \frac{\partial \phi_y}{\partial y} + D_{26} \frac{\partial \phi_y}{\partial x} + D_{16} \frac{\partial \phi_x}{\partial y}, \quad (4b)$$

$$M_y = D_{16} \frac{\partial \phi_x}{\partial x} + D_{26} \frac{\partial \phi_y}{\partial y} + D_{66} \frac{\partial \phi_y}{\partial x} + D_{16} \frac{\partial \phi_x}{\partial y}, \quad (4c)$$

$$Q_x = kA_{55} \left( \phi_x + \frac{\partial w}{\partial x} \right) + kA_{45} \left( \phi_y + \frac{\partial w}{\partial y} \right), \quad (5a)$$

$$Q_y = kA_{45} \left( \phi_x + \frac{\partial w}{\partial x} \right) + kA_{44} \left( \phi_y + \frac{\partial w}{\partial y} \right). \quad (5b)$$

Where  $A_{ij}$  and  $D_{ij}$  are the stretching and bending stiffness,  $k$  is the shear correction factor taken as 5/6. Also, the x-y coordinate plane is located on the mid-plane of the laminate.

### DISCRETE SINGULAR CONVOLUTION (DSC)

Here, the method of discrete singular convolution is briefly presented. Details of the DSC method are not given; interested readers may refer to the studies of Wei (2001a). It is known that, accurate and

efficient numerical approaches for differential equations are of great importance in engineering sciences. The method of Discrete Singular Convolutions (DSC) has emerged as a new approach for numerical solutions of differential equations. This new method has a potential approach for computer realization as a wavelet collocation scheme. For brevity, consider a distribution,  $T$  and  $\eta(t)$  as an element of the space of the test function. A singular convolution can be defined, in the context of distribution theory, by Wei (2001b).

$$F(t) = (T * \eta)(t) = \int_{-\infty}^{\infty} T(t-x)\eta(x)dx \quad (6)$$

where  $T(t-x)$  is a singular kernel. The DSC algorithm can be realized by using many approximation kernels. However, it was shown (Wei, 2001a, b, c) that for many problems, the use of the Regularized Shannon kernel (RSK) is very efficient. The RSK is given by Wei (2001c)

$$\delta_{\Delta, \sigma}(x - x_k) = \frac{\sin[(\pi/\Delta)(x - x_k)]}{(\pi/\Delta)(x - x_k)} \exp\left[-\frac{(x - x_k)^2}{2\sigma^2}\right]; \sigma > 0 \quad (7)$$

where  $\Delta = \pi/(N-1)$  is the grid spacing and  $N$  is the number of grid points. The parameter  $\sigma$  determines the width of the Gaussian envelope and often varies in association with the grid spacing, i.e.,  $\sigma = r\Delta$ . In the DSC method, the function  $f(x)$  and its derivatives with respect to the  $x$  coordinate at a grid point  $x_i$  are approximated by a linear sum of discrete values  $f(x_k)$  in a narrow bandwidth  $[x - x_M, x + x_M]$ . For numerical computations, this can be expressed as:

$$\left. \frac{d^n f(x)}{dx^n} \right|_{x=x_i} = f^{(n)}(x) \approx \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(n)}(x_i - x_k) f(x_k); (n=0, 1, 2, \dots) \quad (8)$$

where superscript  $n$  denotes the  $n$ th-order derivative with respect to  $x$ . The  $x_k$  is a set of discrete sampling points centred around the point  $x$ ,  $\sigma$  is a regularization parameter,  $\Delta$  is the grid spacing and  $2M+1$  is the computational bandwidth, which is usually smaller than the size of the computational domain. It is also known that, it provides exact results when sampling points are extended to an infinite series given by:

$$f(x) = \sum_{k=-\infty}^{\infty} f(x_k) = \frac{\sin[(\pi/\Delta)(x - x_k)]}{(\pi/\Delta)(x - x_k)} \quad \forall f \in B_{\pi/\Delta}^2 \quad (9)$$

The higher order derivative terms  $\delta_{\Delta, \sigma}^{(n)}(x - x_k)$  are given as below:

$$\delta_{\Delta, \sigma}^{(n)}(x - x_k) = \left(\frac{d}{dx}\right)^n [d_{\Delta, \sigma}(x - x_k)] \quad (10)$$

where, the differentiation can be carried out analytically. The discretized forms of Eq. 5 can then be expressed as:

$$f^{(n)}(x) = \left. \frac{d^n f}{dx^n} \right|_{x=x_i} \approx \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(n)}(k\Delta x_N) f_{i+k, j} \quad (11)$$

When the regularized Shannon's delta (RSD) kernel is used, the detailed expressions for,  $\delta_{\Delta, \sigma}^{(n)}(x)$  can be easily obtained. Detailed formulations for these differentiation coefficients can be found in references (Wei, 2001c; Civalek, 2006a). The detailed expressions for  $\delta_{\Delta, \sigma}^{(1)}(x)$  and  $\delta_{\Delta, \sigma}^{(2)}(x)$  and can be easily obtained as (for  $x \neq x_k$ ):

$$\delta_{\pi/\Delta, \sigma}^{(1)}(x_m - x_k) = \frac{\cos(\pi/\Delta)(x - x_k)}{(x - x_k)} \exp[-(x - x_k)^2/2\sigma^2] - \frac{\sin(\pi/\Delta)(x - x_k)}{\pi(x - x_k)^{2/\Delta}} \exp[-(x - x_k)^2/2\sigma^2] \\ - \frac{\sin(\pi/\Delta)(x - x_k)}{(\pi\sigma^{2/\Delta})} \exp[-(x - x_k)^2/2\sigma^2] \quad (12a)$$

$$\delta_{\pi/\Delta, \sigma}^{(2)}(x_m - x_k) = -\frac{(\pi/\Delta)\sin(\pi/\Delta)(x - x_k)}{(x - x_k)} \exp[-(x - x_k)^2/2\sigma^2] \\ - 2\frac{\cos(\pi/\Delta)(x - x_k)}{(x - x_k)^2} \exp[-(x - x_k)^2/2\sigma^2] \\ - 2\frac{\cos(\pi/\Delta)(x - x_k)}{\sigma^2} \exp[-(x - x_k)^2/2\sigma^2] + 2\frac{\sin(\pi/\Delta)(x - x_k)}{\pi(x - x_k)^{3/\Delta}} \exp[-(x - x_k)^2/2\sigma^2] \\ + \frac{\sin(\pi/\Delta)(x - x_k)}{\pi(x - x_k)\sigma^{2/\Delta}} \exp[-(x - x_k)^2/2\sigma^2] \\ + \frac{\sin(\pi/\Delta)(x - x_k)}{\pi\sigma^{4/\Delta}} (x - x_k) \exp[-(x - x_k)^2/2\sigma^2] \quad (12b)$$

At  $x = x_k$ , these derivatives can be written as

$$\delta_{\pi/\Delta, \sigma}^{(1)}(0) = 0 \quad (13a)$$

$$\delta_{\pi/\Delta, \sigma}^{(2)}(0) = -\frac{1}{3} \frac{3 + (\frac{\pi^2}{\Delta^2})\sigma^2}{\sigma^2} = -\frac{1}{\sigma^2} - \frac{\pi^2}{3\Delta^2} \quad (13b)$$

## IMPOSING OF BOUNDARY CONDITIONS

Two types of boundary conditions for an arbitrary edges, i.e., simply supported (S) and clamped (C) are taken into consideration. Following, the related formulations and their DSC form are given in detail.

- Simply supported edge (S)

$$\text{SS1:} \quad w = 0, M_n = 0, M_{ns} = 0 \quad (14a)$$

$$\text{SS2: } w = 0, M_n = 0, \theta_s = 0. \quad (14b)$$

Clamped edge (C)

$$w = 0, \theta_n = 0, \theta_s = 0. \quad (15)$$

where  $n$  and  $s$  denote the normal and tangential directions of the plate, respectively;  $M_n$  and  $M_{ns}$  represent the normal bending and twisting moment;  $\theta_n$  and  $\theta_s$  are rotations about the tangential and normal coordinates at the edge. The force resultants and the rotations on the edge are given as follows (Liu *et al.*, 2002):

$$M_n = n_x^2 M_x + 2n_x n_y M_{xy} + n_y^2 M_y, \quad (16a)$$

$$M_{ns} = (n_x^2 - n_y^2) M_{xy} + n_x n_y (M_y - M_x), \quad (16b)$$

$$\theta_n = n_x \phi_x + n_y \phi_y, \quad (16c)$$

$$\theta_s = n_x \phi_y - n_y \phi_x, \quad (16d)$$

Here  $n_x$  and  $n_y$  are the direction cosines of a unit normal vector at the boundary. Using the force resultants and the rotations, the equations of boundary conditions becomes,

- SS1

$$w = 0 \quad (17a)$$

$$\begin{aligned} M_n = n_x^2 & \left[ D_{11} \frac{\partial \psi_x}{\partial x} + D_{12} \frac{\partial \psi_y}{\partial y} + D_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] \\ & + 2n_x n_y \left[ D_{16} \frac{\partial \psi_x}{\partial x} + D_{26} \frac{\partial \psi_y}{\partial y} + D_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] \\ & + n_y^2 \left[ D_{12} \frac{\partial \psi_x}{\partial x} + D_{22} \frac{\partial \psi_y}{\partial y} + D_{26} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] = 0 \end{aligned} \quad (17b)$$

$$\begin{aligned} M_{ns} = (n_x^2 - n_y^2) & \left[ D_{16} \frac{\partial \psi_x}{\partial x} + D_{26} \frac{\partial \psi_y}{\partial y} + D_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] \\ & + n_x n_y \left[ D_{12} \frac{\partial \psi_x}{\partial x} + D_{22} \frac{\partial \psi_y}{\partial y} + D_{26} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] \\ & - \left[ D_{11} \frac{\partial \psi_x}{\partial x} + D_{12} \frac{\partial \psi_y}{\partial y} + D_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] = 0 \end{aligned} \quad (17c)$$

- SS2

$$w = 0 \quad (18a)$$

$$\begin{aligned} M_n = n_x^2 & \left[ D_{11} \frac{\partial \psi_x}{\partial x} + D_{12} \frac{\partial \psi_y}{\partial y} + D_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] \\ & + 2n_x n_y \left[ D_{16} \frac{\partial \psi_x}{\partial x} + D_{26} \frac{\partial \psi_y}{\partial y} + D_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] \\ & + n_y^2 \left[ D_{12} \frac{\partial \psi_x}{\partial x} + D_{22} \frac{\partial \psi_y}{\partial y} + D_{26} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] = 0 \end{aligned} \quad (18b)$$

$$\theta_s = n_x \psi_y - n_y \psi_x = 0 \quad (18c)$$

- Clamped

$$w = 0 \quad (19a)$$

$$\theta_n = n_x \psi_x + n_y \psi_y \quad (19b)$$

$$\theta_s = n_x \psi_y - n_y \psi_x = 0 \quad (19c)$$

In the method of DSC, Wei *et al.* (2002) proposed a practical method in applying the simply supported and clamped boundary conditions. Recently, a new approach called the iteratively matched boundary method in applying boundary conditions in the DSC method was also proposed by Zhao *et al.* (2005) and applied to free boundary condition of beams. Following the same procedure proposed by Wei *et al.* (2002), consider a uniform grid having following form:

$$0 = X_0 < X_1 < \dots < X_{N_x} = 1 \quad (20a)$$

$$0 = Y_0 < Y_1 < \dots < Y_{N_y} = 1 \quad (20b)$$

Consider a column vector  $W$  given as:

$$W = (W_{0,0}, \dots, W_{0,N}, W_{1,0}, \dots, W_{N,N})^T \quad (21)$$

with  $(N_x + 1)(N_y + 1)$  entries  $W_{ij} = W(X_i, Y_j)$ ; ( $i = 0, 1, \dots, N_x$ ;  $j = 0, 1, \dots, N_y$ ). Let us define the  $(N_x + 1)(N_y + 1)$  differentiation matrices  $D_r^n$  ( $r = X, Y$ ;  $n = 1, 2, \dots$ ), with their elements given by Wei *et al.* (2002)

$$[D_x^{(n)}]_{i,j} = \delta_{\alpha,\Delta}^{(n)}(x_i - x_j) \quad (22a)$$

$$[D_y^{(n)}]_{i,j} = \delta_{\alpha,\Delta}^{(n)}(y_i - y_j) \quad (22b)$$

where  $\delta_{\alpha,\Delta}^{(n)}(r_i - r_j)$ , ( $r = x, y$ ) is a DSC kernel of delta type. For regularized Shannons delta kernel, the differentiation in Eq. 26 can be given by:

$$[D_x^{(n)}]_{i,j} = \delta_{\alpha,\Delta}^{(n)}(x_i - x_j) = \left[ \left( \frac{d}{dx} \right)^n \delta_{\alpha,\Delta}(x - x_j) \right]_{x=x_i} \quad (23a)$$

$$[D_y^{(n)}]_{i,j} = \delta_{\sigma,\Delta}^{(n)}(y_i - y_j) = \left[ \left( \frac{d}{dy} \right)^n \delta_{\sigma,\Delta}(y - y_j) \right]_{y=y_i} \quad (23b)$$

In this stage, we consider the following relation between the inner nodes and outer nodes on the left boundary:

$$W(X_{-i}) - W(X_0) = W(X_0) \left( \sum_{j=0}^J a_i X_{-i} \right) [W(X_i) - W(X_0)] \quad (24)$$

After rearrangement, ones obtain:

$$W(X_{-i}) = a_i W(X_i) + (1 - a_i) W(X_0) \quad (25)$$

where, parameter  $a_i$ , ( $i = 1, 2, \dots, M$ ) are to be determined by the boundary conditions. Thus, the first order derivative of  $W$  on the left boundary are approximated by:

$$\begin{aligned} W'(X_0) = & \left( \delta_{\sigma,\Delta}^{(1)}(X_i - X_0) - \sum_{j=0}^J (1 - a_i) \delta_{\sigma,\Delta}^{(1)}(X_i - X_j) \right) W(X_0) \\ & + \sum_{j=0}^J (1 - a_i) \delta_{\sigma,\Delta}^{(1)}(X_i - X_j) W(X_i) \end{aligned} \quad (26)$$

Similarly, the first order derivative of  $f$  on the right boundary (at  $X_{N-1}$ ) are approximated by:

$$W(X_{N-1+i}) - W(X_{N-1}) = a_i [W(X_{N-1-i}) - W(X_{N-1})], \quad (27)$$

or

$$W(X_{N-1+i}) - W(X_{N-1}) = W(X_{N-1-i}) \left( \sum_{j=0}^J a_i X_{-i} \right) [W(X_i) - W(X_N)] \quad (28)$$

Consequently, we obtain the following relation:

$$W(X_{N-1+i}) = a_i W(X_{N-1-i}) + W(X_{N-1}) [1 - a_i]. \quad (29)$$

Hence, the first order derivative of  $f$  on the right boundary is given by :

$$\begin{aligned} W'(X_{N-1}) = & \left( \delta_{\sigma,\Delta}^{(1)}(X_i - X_{N-1}) - \sum_{j=0}^J (1 - a_i) \delta_{\sigma,\Delta}^{(1)}(X_i - X_j) \right) W(X_{N-1}) \\ & + \sum_{j=0}^J (1 - a_i) \delta_{\sigma,\Delta}^{(1)}(X_i - X_j) W(X_i) \end{aligned} \quad (30)$$

Only the governing Eq. 1c is used for buckling, that is:



$$\begin{aligned}
& \frac{\partial}{\partial x} \left[ kA_{45} \left( \varphi_y + \frac{\partial w}{\partial y} \right) + kA_{55} \left( \varphi_x + \frac{\partial w}{\partial x} \right) \right] \\
& + \frac{\partial}{\partial y} \left[ kA_{44} \left( \varphi_y + \frac{\partial w}{\partial y} \right) + kA_{55} \left( \varphi_x + \frac{\partial w}{\partial x} \right) \right] \\
& + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0.
\end{aligned} \tag{31}$$

According to the DSC method, the governing equations Eq. 1c can be discretized into the following form for buckling:

$$\begin{aligned}
& kA_{45} \left( \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(1)}(k\Delta x) \psi_{kj}^y + \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(1)}(k\Delta x) \psi_{kj}^x + 2 \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(1)}(k\Delta x) W_{kj} \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(1)}(k\Delta y) W_{ik} \right) \\
& kA_{55} \left( \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(1)}(k\Delta x) \psi_{kj}^x + \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(2)}(k\Delta x) W_{kj} \right) + kA_{44} \left( \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(1)}(k\Delta y) \psi_{ik}^y + \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(2)}(k\Delta y) W_{ik} \right) \\
& + N_x \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(2)}(k\Delta x) W_{kj} + 2N_{xy} \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(1)}(k\Delta x) W_{kj} \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(1)}(k\Delta y) W_{ik} + N_y \sum_{k=-M}^M \delta_{\Delta, \sigma}^{(2)}(k\Delta y) W_{ik} = 0
\end{aligned} \tag{32}$$

Similarly, DSC form of the related boundary conditions can also be given. Discretized form of SS2 boundary conditions, for example, are given by:

$$w_{ij} = 0 \tag{33a}$$

$$\begin{aligned}
& n_x^2 \left\{ D_{11} \left[ \left( \delta_{\sigma, \Delta}^{(1)}(X_i - X_0) - \sum_{j=0}^J (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(X_i - X_j) \right) \psi(X_0) + \sum_{j=0}^J (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(X_i - X_j) \psi(X_j) \right] \right. \\
& + D_{12} \left[ \left( \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_0) - \sum_{k=0}^K (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_k) \right) \psi(Y_0) + \sum_{k=0}^K (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_k) \psi(Y_k) \right] \\
& + D_{16} \left[ \left( \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_0) - \sum_{k=0}^K (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_k) \right) \psi(X_0) + \sum_{k=0}^K (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_k) \psi(X_i) \right] \\
& + D_{16} \left[ \left( \delta_{\sigma, \Delta}^{(1)}(X_i - X_0) - \sum_{j=0}^J (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(X_i - X_j) \right) \psi(Y_0) + \sum_{j=0}^J (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(X_i - X_j) \psi(Y_i) \right] \Big\} \\
& + 2n_x n_y \left\{ D_{16} \left[ \left( \delta_{\sigma, \Delta}^{(1)}(X_i - X_0) - \sum_{j=0}^J (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(X_i - X_j) \right) \psi(X_0) + \sum_{j=0}^J (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(X_i - X_j) \psi(X_j) \right] \right. \\
& + D_{26} \left[ \left( \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_0) - \sum_{k=0}^K (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_k) \right) \psi(Y_0) + \sum_{k=0}^K (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_k) \psi(Y_k) \right] \\
& + D_{66} \left[ \left( \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_0) - \sum_{k=0}^K (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_k) \right) \psi(X_0) + \sum_{k=0}^K (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(Y_i - Y_k) \psi(X_i) \right] \\
& + D_{66} \left[ \left( \delta_{\sigma, \Delta}^{(1)}(X_i - X_0) - \sum_{j=0}^J (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(X_i - X_j) \right) \psi(Y_0) + \sum_{j=0}^J (1 - a_i) \delta_{\sigma, \Delta}^{(1)}(X_i - X_j) \psi(Y_i) \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& +n_y^2 \left\{ D_{12} \left[ \left( \delta_{\sigma,\Delta}^{(1)}(X_1 - X_0) - \sum_{j=0}^J (1-a_i) \delta_{\sigma,\Delta}^{(1)}(X_1 - X_j) \right) \psi(X_0) + \sum_{j=0}^J (1-a_i) \delta_{\sigma,\Delta}^{(1)}(X_1 - X_j) \psi(X_1) \right] \right. \\
& + D_{22} \left[ \left( \delta_{\sigma,\Delta}^{(1)}(Y_1 - Y_0) - \sum_{k=0}^K (1-a_i) \delta_{\sigma,\Delta}^{(1)}(Y_1 - Y_k) \right) \psi(Y_0) + \sum_{k=0}^K (1-a_i) \delta_{\sigma,\Delta}^{(1)}(Y_1 - Y_k) \psi(Y_1) \right] \\
& + D_{26} \left[ \left( \delta_{\sigma,\Delta}^{(1)}(Y_1 - Y_0) - \sum_{k=0}^K (1-a_i) \delta_{\sigma,\Delta}^{(1)}(Y_1 - Y_k) \right) \psi(X_0) + \sum_{k=0}^K (1-a_i) \delta_{\sigma,\Delta}^{(1)}(Y_1 - Y_k) \psi(X_1) \right] \\
& \left. + D_{26} \left[ \left( \delta_{\sigma,\Delta}^{(1)}(X_1 - X_0) - \sum_{j=0}^J (1-a_i) \delta_{\sigma,\Delta}^{(1)}(X_1 - X_j) \right) \psi(Y_0) + \sum_{j=0}^J (1-a_i) \delta_{\sigma,\Delta}^{(1)}(X_1 - X_j) \psi(Y_1) \right] \right\} = 0
\end{aligned} \quad (33b)$$

$$n_x \psi_{ik} - n_y \psi_{ij} = 0 \quad (33c)$$

Consequently, we solve the remaining eigenvalue problems given below to obtain the non-dimensional buckling load, such as,

$$GX = \lambda BX \quad (34)$$

where X is the displacement vector defined as follows:

$$X = [W_{ij} \quad \psi_{ij}]^T \quad (35)$$

In Eq. 33, G and B are the matrices derived from the governing equations and the boundary conditions. In the above eigenvalue equations,  $\lambda$  is the non-dimensional buckling load.

## NUMERICAL EXAMPLES

In numerical solutions of laminate are assumed to be of the same thickness and density. Linearly elastic composite material behavior is taken into consideration. In all the tables, S denotes simply supported while C means clamped. The notation, for example, SCSC denotes a plate having simply supported with edges  $y = 0$  and  $y = b$  and having clamped with edges  $x = 0$  and  $x = a$ . Following values for material parameters are used for numerical analysis.

$$G_{12} = G_{13} = 0.6E_2; G_{23} = 0.5E_2; \nu_{12} = 0.25; E_1/E_2 = 40$$

Several examples are solved and results are presented in Table 1-5. For comparison purpose, uniaxially buckling loads of a SSSS laminated ( $0^\circ/90^\circ/90^\circ/0^\circ$ ) square plate is obtained by the DSC method using the 15 grid points. The results in Table 1 are compared respectively to the analytical

Table 1: Comparisons of uniaxially buckling loads of a SSSS laminated ( $0^\circ/90^\circ/90^\circ/0^\circ$ ) square plate ( $a/h=10$ ;  $\lambda = N_x a^2/E_2 h^3$ )

Sources				
$E_1/E_2$	HOSDT		FOSDT	Present study
	Noor (1975)	Khdeir and Librescu (1988)	Khdeir and Librescu (1988)	
20	15.0191	15.418	15.351	15.348
30	19.3040	19.813	19.757	19.756
40	22.8807	23.489	23.453	23.452

Table 2: Non-dimensional biaxial buckling loads of laminated ( $0^\circ/90^\circ/0^\circ$ ) square plate ( $a/h=10$ ;  $\lambda = N_x a^2/E_2 h^3$ ;  $N_x = N_y$ ) for different boundary conditions

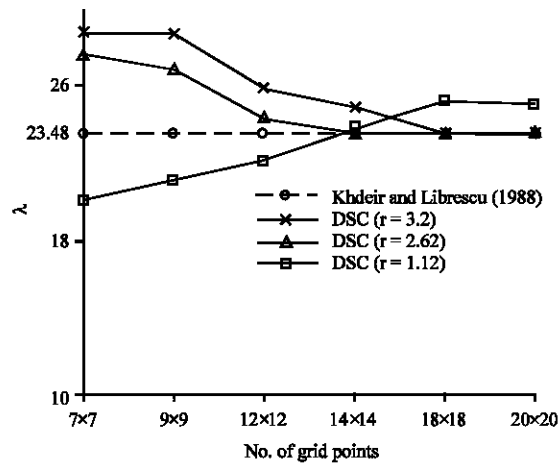
$E_1/E_2$	Boundary conditions		
	SSSS	SSCS	CSCS
20	7.497	9.062	10.758
30	9.043	10.470	12.177
40	10.241	11.632	13.286

Table 3: Non-dimensional biaxial buckling loads of laminated ( $0^\circ/90^\circ/0^\circ$ ) square plate ( $\lambda = N_x a^2/E_2 h^3$ ;  $N_x = N_y$ ) for different thickness ratio

$a/h$	Boundary conditions		
	SSSS	SSCS	CSCS
2	1.429	1.434	1.458
5	5.495	5.879	6.112
9	9.968	11.516	13.004
15	12.187	15.451	19.595

Table 4: Non-dimensional biaxial buckling loads of laminated ( $0^\circ/90^\circ/0^\circ$ ) square plate ( $a/h = 10$ ;  $E_1/E_2 = 40$ ;  $\lambda = N_x a^2/E_2 h^3$ ;  $N_x = N_y$ ) for different aspect ratio

$b/a$	Boundary conditions		
	SSSS	SSCS	CSCS
1	10.241	11.632	13.286
2	21.198	22.020	23.455

Fig. 1: Convergence of the buckling load of SSSS laminated ( $0^\circ/90^\circ/90^\circ/0^\circ$ ) square plate ( $a/h=10$ ;  $E_1/E_2 = 40$ ;  $\lambda = N_x a^2/E_2 h^3$ ;

solutions based on First-order Shear Deformation Theory (FOSDT) and higher-order shear deformation theory by Khdeir and Librescu (1988), the three-dimensional linear elasticity solutions of Noor (1975). Compared with the data given by Khdeir and Librescu (1988), it is shown that the present results are in close agreement.

Convergence of the buckling load of SSSS laminated ( $0^\circ/90^\circ/90^\circ/0^\circ$ ) square plate is depicted in Fig. 1 with the different parameter  $r$  and different value of grid numbers are shown. For this purpose, the results given by Khdeir and Librescu (1988) for  $E_1/E_2 = 40$  is used listed in Table 1. It is shown in this figure that the reasonable accurate results are obtained for  $N > 14$ . Also, from Fig. 1, we can see that the optimal convergence could be achieved with the value of between  $2.4 < r < 3.2$ . It was also

Table 5: Non-dimensional biaxial buckling loads of laminated ( $0^\circ/90^\circ/90^\circ/0^\circ$ ) square plate ( $\lambda = N_x a^2/E_2 h^3$ ;  $N_x = N_y$ ) for different value of material and geometric parameters

a/h	$E_1/E_2$			
	3	10	20	40
100	5.7603	11.4788	19.6670	35.9554
80	5.7543	11.4615	19.5861	35.7820
50	5.7411	11.4206	19.4884	35.3573
10	5.4117	9.9485	15.3482	23.4524
5	4.5561	7.1659	9.4323	12.0981

shown (Civalek, 2006b; Wei, 2001b; Lim *et al.*, 2005) that the parameter  $r$  gives more accurate results for the interval  $2.2 \leq r \leq 3$  in applied mechanics. Thus, during the study we set the parameter  $r$  as 2.62 and  $N = 16$ .

Non-dimensional biaxial buckling loads of laminated ( $0^\circ/90^\circ/0^\circ$ ) square plate for various values of orthotropy of individual layers  $E_1/E_2$  and different boundary conditions are presented in Table 2. With increase of ratio  $E_1/E_2$ , the buckling loads increases relatively. Non-dimensional biaxial buckling loads of laminated ( $0^\circ/90^\circ/0^\circ$ ) square plate are listed in Table 3. Four different thickness-side ratio  $h/a$  are used. Also, results are presented for different boundary conditions. Non-dimensional biaxial buckling loads of laminated ( $0^\circ/90^\circ/0/20$ ) square plate are presented in Table 4 and Table 5 for different geometric and orthotropic properties. From the results obtained it can be possible to say that the DSC method can be used for solving vibration and buckling problems of laminated plates.

## CONCLUSIONS

Buckling loads of laminated composite plates are obtained using the discrete singular convolution method. The first-order shear deformation theory (FSDT) is used in the study with the governing differential equations transformed into a standard eigenvalue problem by the discrete singular convolution formulation. The results are obtained for different geometric and material parameters for various combinations of simply supported and clamped boundary conditions. The accuracy of the proposed method is confirmed with the available numerical and analytical solutions. It is concluded that the DSC method provides accurate solutions.

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