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Adaptive Coupled Synchronization of Coupled Chaotic Dynamical Systems

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Abstract: In this study, the adaptive synchronization method of coupled system is applied to achieve synchronization for hyperchaotic Lü system and coupled van der Pol oscillators. This method can avoid estimating the value of coupling coefficient. Lyapunove direct method of stability is used to prove the asymptotic stability of solutions for the error dynamical system. Numerical simulations results are used to demonstrate the effectiveness of the proposed control strategy.

Key words: Hyperchaotic Lü system, coupled van der Pol oscillators, adaptive synchronization, lyapunove function, numerical simulation

INTRODUCTION

Deterministic chaos has been thoroughly investigated in the last three decades since it was found that many realworld physical systems could behave chaotically (Strogatz, 1994; Femat and Alvarez-Ramirez, 1997). It has been found that chaos may be useful in many fields (Hwang *et al*., 1996; Femat *et al*., 2001). However, it has also appeared that on another hand, chaos may be undesirable in some cases where regular oscillations are needed, like metal cutting processes (Wiercigroch and Krivtsov, 2001), power electronics (Chen *et al*., 1999) and so on.

In recent years, researches on chaos control and synchronization have attracted increasing attention due to its potential applications to physics, chemical reactors, control theories, biological networks, artificial neural networks and secure communication (Chen and Dong, 1998; Pyragas, 1992; Tao *et al*., 2005; Wang and Tian, 2004).

In 1989, Hubler published the first article on chaos control. Ott *et al*. (1990), developed the OGY method (Ott *et al*., 1990). In the same year, Pecora and Carroll (1990) and Corral and Perca (1991) proposed the idea of chaos synchronization. In the past ten years, many techniques for chaos control and synchronization have been developed, such as feedback method, adaptive technique, time delay feedback approach, backstepping method and so on.

Hyperchaotic systems is usually classified as a chaotic system with more than one positive Lyapunov exponent, indicating that the chaotic dynamics of the system are expanded in more than one direction giving rise to a more complex attractor. In recent years, hyperchaos has been studied with increasing interests, in the fields of secure communication (Udaltsov *et al*., 2003), multimode lasers (Shahverdiev *et al*., 2004), nonlinear circuits (Barbara and Silvano, 2002), biological networks (Neiman *et al*., 1999), coupled map lattices (Zhan *et al*., 2000) and so on.

SYSTEM DESCRIPTION

In this paper we study the synchronization of the hyperchaotic Lü system (Elabbasy *et al*., 2006) and synchronization between the hyperchaotic Lü system and coupled van der Pol oscillators (Fotsin and Woafo, 2005).

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Hyperchaotic Lü System

The hyperchaotic Lü system is described by the following system of differential equations:

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= -xz + cy + w \\ \dot{z} &= xy - bz \\ \dot{w} &= z - rw \end{aligned} \tag{1}$$

where a, b, c and r are four unknown uncertain parameters. This new system exhibits a chaotic attractor at the parameter values a = 15, b = 5, c = 10 and r = 1 (Fig. 1a and b).

The divergence of the flow (1) is given by

$$\nabla . F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} + \frac{\partial F_4}{\partial w} = -a + c - b - r < 0.$$

where $F = (F_1, F_2, F_3, F_4) = (a(y - x),\ -xz + cy + w,\ xy - bz,\ z - rw)$

Hence the system is dissipative when: c<a+b+c

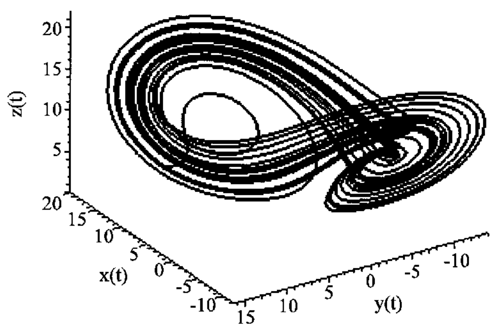

Fig. 1a: Shows the chaotic attractor of hyperchaotic Lü system at a = 15, b = 5, c = 10 and r = 1 in x, y, z subspace

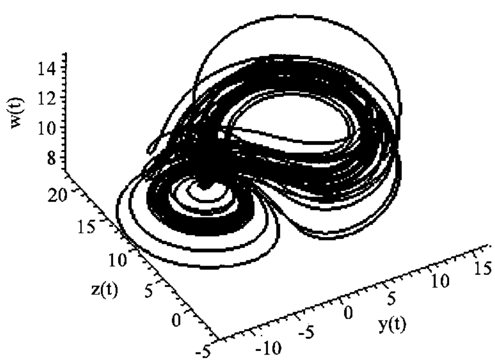

Fig. 1b: Show the chaotic attractor of hyperchaotic Lü system at a = 15, b = 5, c = 10 and r = 1 in y, z, w subspace

The system has three equilibrium points:

$$E_0 = (0,0,0),\ E_+ = (\sigma_1, \sigma_1, \frac{\sigma_1^2}{b}, \frac{\sigma_1^2}{br}),\ E_- = (\sigma_2, \sigma_2, \frac{\sigma_2^2}{b}, \frac{\sigma_2^2}{br})$$

where $\sigma_1 = \frac{1+\sqrt{1+4bcr^2}}{2d}$ and $\sigma_2 = \frac{1-\sqrt{1+4bcr^2}}{2d}$

To study the stability of E_0 the associated Jacobian J_0 is

$$J_0 = \begin{bmatrix} -a & a & 0 & 0 \\ -z & c & -x & 1 \\ y & x & -b & 0 \\ 0 & 0 & 1 & -r \end{bmatrix}$$

The characteristic polynomial of the matrix J_0 is given by

$$(\lambda + a)(\lambda - c)(\lambda + b)(\lambda + r) = 0 \tag{2}$$

The eigenvalues are λ_1 = -a, λ_2 = c, λ_3 = -b and λ_4 = -r. Then the equilibrium point E_0 is stable if c<0 otherwith the equlibrium is unstable.

To study the stability of E_0 the associated Jacobian j_+ is

$$J_+ = \begin{bmatrix} -a & a & 0 & 0 \\ -\frac{2cbr^2+1+\sqrt{1+4cbr^2}}{2br^2} & c & -\frac{1+\sqrt{1+4cbr^2}}{2r} & 1 \\ \frac{1+\sqrt{1+4cbr^2}}{2r} & \frac{1+\sqrt{1+4cbr^2}}{2r} & -b & 0 \\ 0 & 0 & 1 & -r \end{bmatrix}$$

The characteristic polynomial of the matrix j_+ is given by

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0 \tag{3}$$

where

$$c_1 = r + b - c + a$$

$$c_2 = \frac{a + b + 2b^2r^3 + (a+b)\sqrt{1+4cbr^2} - 2br^3c + 2abr^3 + 2ab^2r^3}{2br^2}$$

$$c_3 = \frac{3ab + ar + 2ab^2r^3 + (ar + 3ab)\sqrt{1+4cbr^2} + 4acb^2r^2}{2br^2}$$

$$c_4 = \frac{a + 4abcr^2 + a\sqrt{1+4cbr^2}}{2r}$$

A set of necessary and sufficient conditions for all the roots of Eq. 3 to have negative real parts is given by the well-known Routh-Hurwitz criterion in the following form

$$c_1 > 0,\ c_1c_2 - c_3 > 0,\ c_1(c_2c_3 - c_1c_4) - c_3^2 > 0 \text{ and } c_1c_4(c_2c_3 - c_1c_4) - c_4c_3^2 > 0$$

i.e.,

$$c_1 > 0,\ \ c_4 > 0,\ c_1c_2 - c_3 > 0 \text{ and } c_1(c_2c_3 - c_1c_4) - c_3^2 > 0$$

However, the above values of c_1, c_4 and c_3 guaranteed that $c_1c_2 - c_3 < 0$. Hence the equilibrium point E_+ is unstable.

To study the stability of E_ the associated Jacobian J_ is

$$J\_ = \begin{bmatrix} -a & a & 0 & 0 \\ -\dfrac{2cbr^2 + 1 - \sqrt{1+4cbr^2}}{2br^2} & c & -\dfrac{1-\sqrt{1+4cbr^2}}{2r} & 1 \\ \dfrac{1-\sqrt{1+4cbr^2}}{2r} & \dfrac{1-\sqrt{1+4cbr^2}}{2r} & -b & 0 \\ 0 & 0 & 1 & -r \end{bmatrix}$$

The characteristic polynomial of the matrix J_ is given by

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0 \tag{4}$$

where

$$c_1 = r + b - c + a$$

$$c_2 = \frac{a + b + 2b^2r^3 - (a+b)\sqrt{1+4cbr^2} - 2br^3c + 2abr^3 + 2ab^2r^3}{2br^2}$$

$$c_3 = \frac{3ab + ar + 2ab^2r^3 - (ar + 3ab)\sqrt{1+4cbr^2} + 4acb^2r^2}{2br^2}$$

$$c_4 = \frac{a + 4abcr^2 - a\sqrt{1+4cbr^2}}{2r}$$

As above, one can see that E_ is also unstable since $c_1c_2 - c_3$ will be negative.

The Coupled Van Der Pol Oscillators

The circuit diagram of the chaotic coupled van der Pol oscillators (Fotsin and Woafo, 2005) is shown in Fig. 2. The nonlinear resistance (NR) part is characterized by a third-order voltagecurrent (iV) characteristic of the form $i(V) = aV + bV_3$ ($a<0$, $b>0$). An electronic circuit for such a resistance can be found in (Fotsin and Woafo, 2005).

It can easily be shown (Fotsin and Woafo, 2005) that the dynamics of the circuit of Fig. 2 is described by the following set of coupled second-order differential equations:

$$\begin{aligned} &\ddot{V}_1 + \frac{a}{C_2}\left[1 + \frac{3b}{a}V_1^2\right]\dot{V}_1 + \frac{1}{L_2C_2}V_1 + \frac{C_1}{C_2}\ddot{V}_2 = 0 \\ &\ddot{V}_2 + \frac{R}{L_1}\dot{V}_2 + \frac{1}{L_1C_1}V_2 - \frac{1}{L_1C_1}V_1 = 0 \end{aligned} \tag{5}$$

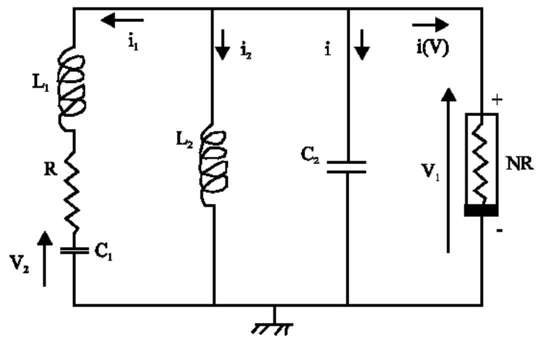

Fig. 2: Show the circuit diagram of the coupled van der Pol oscillators

where the over dot denotes the differentiation with respect to the time t. Setting

$$\begin{aligned} &t = \tau\sqrt{L_2 C_2};\; x_1 = V_1\sqrt{\frac{-3b}{a}};\; \varepsilon_1 = -a\sqrt{\frac{L_2}{C_2}};\; \varepsilon_2 = \frac{R}{L_1}\sqrt{L_2 C_2}, \\ &\alpha = \frac{C_1}{C_2}\sqrt{\frac{-3b}{a}};\; \omega_1^2 = \frac{L_2 C_2}{L_1 C_1};\; \lambda = \omega_1^2\sqrt{\frac{-3b}{a}};\; x_3 = V_2 \end{aligned} \tag{6}$$

The system of Eq. 5 can be rewritten in the following form:

$$\begin{aligned} &\ddot{x}_1 - \varepsilon_1(1-x_1^2)\dot{x}_1 + x_1 + \alpha\,\ddot{x}_3 = 0 \\ &\ddot{x}_3 + \varepsilon_2\dot{x}_3 + \omega_1^2\, x_3 - \lambda\, x_1 = 0 \end{aligned} \tag{7}$$

where the overdot now indicates the differentiation with respect to s (that we rename as t in the new scale without loss of generality). Obviously the system of Eq. 7 represent a van der Pol oscillator (x_1) coupled to the linear oscillator (x_2).

If in addition we set $\dot{x}_1 = x_2$ and $\dot{x}_3 = x_4$, the system of Eq. 7 can now take the following general form:

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = \varepsilon_1(1-x_1^2)x_2 - x_1 + \alpha(\varepsilon_2 x_4 + \omega_1^2 x_3 - \lambda\, x_1) \\ &\dot{x}_3 = x_4 \\ &\dot{x}_4 = -\varepsilon_2 x_4 - \omega_1^2 x_3 + \lambda\, x_1 \end{aligned} \tag{8}$$

With the selection of parameters $\varepsilon_1 = 3.872$, $\varepsilon_2 = 0.000645$, $\lambda = 9.12$, $\alpha = 0.457$ and $\omega^2 = 5$ the system shows a chaotic behavior characterized by a maximal Lyapunov exponent $\lambda_{max} = 0.062$ which confirms occurrence of chaotic oscillations. This system exhibits a chaotic attractor at the parameter values, $\varepsilon_1 = 3.872$, $\varepsilon_2 = 0.000645$, $\lambda = 9.12$, $\alpha = 0.457$ and $\omega^2 = 5$ (Fig. 3a and b).

It is easily shown that system (8) has only one equilibrium at the origin (0, 0, 0, 0) where the Jacobian matrix of system (8) is

$$J_{-} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\varepsilon_1 x_1 x_2 - 1 - \alpha\lambda & \varepsilon_1 & \alpha\,\omega_1^2 & \alpha\,\varepsilon_2 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & -\omega_1^2 & -\varepsilon_2 \end{bmatrix}$$

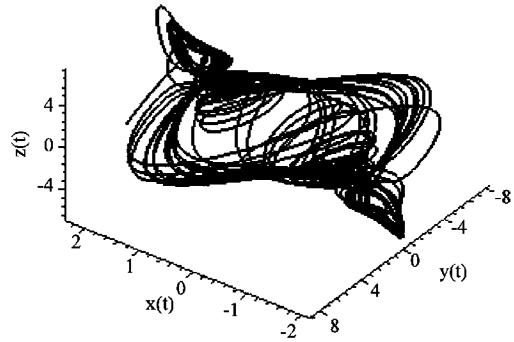

Fig. 3a: Show the chaotic attractor of coupled van der Pol oscillators at $\varepsilon_1 = 3.872$, $\varepsilon_2 = 0.000645$, $\lambda = 9.12$, $\alpha = 0.457$ and $\omega^2 = 5$ in x, y, z subspace

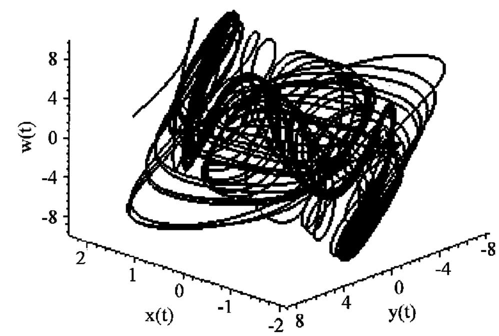

Fig. 3b: Show the chaotic attractor of coupled van der Pol oscillators at $\varepsilon_1 = 3.872$, $\varepsilon_2 = 0.000645$, $\lambda = 9.12$, $\alpha = 0.457$ and $\omega^2 = 5$ in x, y, w subspace

The characteristic polynomial of the matrix J is given by

$$\theta^4 + (\varepsilon_1 - \varepsilon_2)\theta^3 + (1 + \omega_1^2 - \varepsilon_1\ \varepsilon_2 + \alpha\ \lambda)\theta^2 + (\varepsilon_2 - \varepsilon_1\omega_1^2)\theta + \omega_1^2 = 0$$

For the parameters provided above, the first condition of the Routh Hurwitz determinant (which is ε_2-ε_1) is negative. Hence the origin is an unstable equilibrium.

ADAPTIVE SYNCHRONIZATION BETWEEN TWO COUPLED HYPERCHAOTIC LÜ SYSTEM

In order to observe the adaptive synchronization behaviour in hyperchaotic Lü system, we have two hyperchaotic Lü systems where the drive system with four state variables denoted by the subscript 1 drives the response system having identical equations denoted by the subscript 2. However, the initial condition of the drive system is different from that of the response system, therefore two hyperchaotic Lü systems are described, respectively, by the following equations:

$$\begin{aligned} \dot{x}_1 &= a(y_1 - x_1) \\ \dot{y}_1 &= -x_1 z_1 + c y_1 + w_1 + d_1(y_2 - y_1) \\ \dot{z}_1 &= x_1 y_1 - b z_1 + d_2(z_2 - z_1) \\ \dot{w}_1 &= z_1 - r w_1 \end{aligned} \tag{9}$$

and

$$\begin{aligned} \dot{x}_2 &= a(y_2 - x_2) \\ \dot{y}_2 &= -x_2 z_2 + c y_2 + w_2 + d_1 (y_1 - y_2) \\ \dot{z}_2 &= x_2 y_2 - b z_2 + d_2 (z_1 - z_2) \\ \dot{w}_2 &= z_2 - r w_2 \end{aligned} \tag{10}$$

Remark 1

The hyperchaotic Lü system is dissipative system and has a bounded, zero volume, globally attracting set. Therefore, the state trajectories $x_1(t)$, $y_1(t)$, $z_1(t)$ and $w_1(t)$ are globally bounded for all $t \geq 0$ and continuously differentiable with respect to time. Consequently, there exist three positive constants s_1, s_2, s_3 and s_4 such that $|x_1(t)| \leq s_1 < \infty$, $|y_1(t)| \leq s_2 < \infty$, $|z_1(t)| \leq s_3 < \infty$ and $|w_1(t)| \leq s_4 < \infty$ hold for all $t \geq 0$.

Let us define the state errors between the response system that is to be controlled and the controlling drive system as

$$e_x = x_2 - x_1,\ e_y = y_2 - y_1,\ e_z = z_2 - z_1 \ \text{ and } e_w = w_2 - w_1$$

Then the error dynamical system can be written as:

$$\begin{aligned} \dot{e}_x &= a(e_y - e_x) \\ \dot{e}_y &= (c - 2d_1)e_y - x_1 e_z - z_2 e_x + e_w \\ \dot{e}_2 &= y_2 e_x + x_1 e_y - (b + 2d_2) e_z \\ \dot{e}_w &= e_z - r e_w \end{aligned} \tag{11}$$

suppose that

$$\dot{d}_1 = k_1 e_y^2 \ \text{ and } \dot{d}_2 = k_2 e_z^2 \ \text{ such that } k_1,\ k_2 > 0 \tag{12}$$

If $e_x(t) \to 0$, $e_y(t) \to 0$, $e_z(t) \to 0$ and $e_w(t) \to 0$ as $t \to \infty$ the coupling synchronization of the drive system and response system is achieved

Theorem 1

System (9) and (10) will synchronize for any initial values of $(x_1(0),\ y_1(0),\ z_1(0),\ w_1(0))$, $(x_2(0),\ y_2(0),\ z_2(0),\ w_2(0))$, $(d_1(0),\ d_2(0))$ That is $e_x(t) \to 0$, $e_y(t) \to 0$, $e_z(t) \to 0$ and $e_w(t) \to 0$ as $t \to \infty$.

Proof

Consider a Lyapunov function as follows

$$V(e_x, e_y, e_z, e_w, d_1, d_2) = \frac{1}{2}(e_x^2 + e_y^2 + e_z^2 + e_w^2 + \frac{2}{k_1}(d_1 - d_1^*)^2 + + \frac{2}{k_2}(d_2 - d_2^*)^2) \tag{13}$$

where d^*_1 and d^*_2 are positive constants which will be defined later. Taking the time derivative of Eq. 13, then we get

$$
\begin{aligned}
\dot{V} &= e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z + e_w\dot{e}_w + \frac{2}{k_1}(d_1 - d_1^*)\dot{d}_1 + +\frac{2}{k_2}(d_2 - d_2^*)\dot{d}_2 \\
&= -ae_x^2 + ae_xe_y + (c - 2d_1)e_y^2 - z_2e_xe_y - x_1e_ze_y + e_we_y + y_2e_xe_z \\
&\quad + x_1e_ye_z - (b + 2d_2)e_z^2 + e_ze_w - re_w^2 + 2(d_1 - d_1^*)e_y^2 + 2(d_2 - d_2^*)e_z^2 \\
&= -ae_x^2 + (c - 2d_1^*)e_y^2 - (b + 2d_2^*)e_z^2 - re_w^2 + (a - z_2)e_xe_y + y_2e_xe_z + e_ye_w + e_ze_w \\
&\le -ae_x^2 + (c - 2d_1^*)e_y^2 - (b + 2d_2^*)e_z^2 - re_w^2 + (a + s_3)e_xe_y + s_2e_xe_z + e_ye_w + e_ze_w \\
&= -[ae_x^2 + (2d_1^* - c)e_y^2 + (b + 2d_2^*)e_z^2 + re_w^2 - (a + s_3)e_xe_y - s_2e_xe_z - e_ye_w - e_ze_w] \\
&= -\begin{bmatrix}|e_x| & |e_y| & |e_z| & |e_w|\end{bmatrix}
\begin{bmatrix}
a & -\frac{s_3 + a}{2} & -\frac{s_2}{2} & 0 \\
-\frac{s_3 + a}{2} & 2d_1^* - c & 0 & -\frac{1}{2} \\
-\frac{s_2}{2} & 0 & b + 2d_2^* & -\frac{1}{2} \\
0 & -\frac{1}{2} & -\frac{1}{2} & r
\end{bmatrix}
\begin{bmatrix}|e_x| \\ |e_y| \\ |e_z| \\ |e_w|\end{bmatrix} \\
&= -\begin{bmatrix}|e_x| & |e_y| & |e_z| & |e_w|\end{bmatrix}\Psi(d_1^*, d_2^*)\begin{bmatrix}|e_x| & |e_y| & |e_z| & |e_w|\end{bmatrix}^T
\end{aligned} \tag{14}
$$

If

$$
\begin{aligned}
& d_1^* > \frac{a + 2s_3 + 4c + s_3^2}{8} \\
& (4ab - s_2^2)d_1^* + (8ad_1^* - 4ac - 2s_3a - a^2 - s_3^2)d_2^* > \frac{4abc + bs_3^2 + 2s_3ab + a^2b - s_2^2c}{2} \\
& d_1^*\left(\frac{4abr - s_2^2r - a}{2}\right) - d_2^*\left(\frac{4acr - r(s_3 + a)^2 - a}{2}\right) + 4ard_1^*d_2^* > abcr + \frac{a(b - c)}{4} + \frac{a(s_2 - s_3) + s_2s_3}{8} \\
& \qquad + \frac{r(b(s_3 - a)^2 - s_2^2c)}{4} - \frac{a^2 + s_2^2 + s_3^2}{16}
\end{aligned}
$$

hold then the 4×4 matrix $\mathbf{\Psi}(d^*_1, d^*_2)$ is positive definite.

Where s_2 and s_3 are defined in remark 1. If d^*_1 and d^*_2 are appropriately chosen such that the 4×4 matrix $\mathbf{\Psi}(d^*_1, d^*_2)$ in Eq. 14 is positive definite, then $\dot{V} \le 0$ holds. Since V is a positive and decreasing function and $\dot{V}$ is negative semidefinite. It follows that the equilibrium point $e_x = 0, e_y = 0, e_z = 0,$ $e_w = 0, d_1 = d_1^*$ and $d_2 = d_2^*$ of the system (12) is uniformly stable, i.e., $e_x(t), e_y(t), e_z(t), e_w(t) \in L_\infty$ and $d_1(t), d_2(t) \in L_\infty$. From Eq. 14 we can easily show that the squares of $e_x(t), e_y(t), e_z(t)$ and $e_w(t)$ are integrable with respect to time t, i.e., $e_x(t), e_y(t), e_z(t), e_w(t) \in L_2$. Next by Barbalat's Lemma Eq. 12 implies that $\dot{e}_x(t), \dot{e}_y(t), \dot{e}_z(t), \dot{e}_w(t) \in L_\infty$, which in turn implies $e_x(t) \to 0, e_y(t) \to 0, e_z(t) \to 0$ and $e_w(t) \to 0$ as $t \to \infty$ Thus, in the closed-loop system $x_2(t) \to x_1(t), y_2(t) \to y_1(t), z_2(t) \to z_1(t), w_2(t) \to w_1(t)$ as $t \to \infty$. This implies that the two hyperchaotic Lü systems have been globally asymptotically synchronized.

Numerical Results

By using the mathematical package Maple we solve the Eq. 9, 10 and 12. The four parameters are chosen as a = 15, b = 5, c = 10 and r = 1 in all simulations so that the hyperchaotic Lü system exhibits a chaotic behaviour if no control is applied. The initial states of the drive system are $x_1(0) = -20, y_1(0) = 5, z_1(0) = 0$ and $w_1(0) = 15$ and of the response system are $x_2(0) = 10, y_2(0) = -5,$ $z_2(0) = 5$ and $w_2(0) = 10$. Then $e_x(0) = 30, e_y(0) = -10, e_z(0) = 5$ and $e_w(0) = -5$. From the Fig. 1 it can be seen that the solutions x(t), y(t), z(t) and w(t) are bounded and satisfy the inequalities:

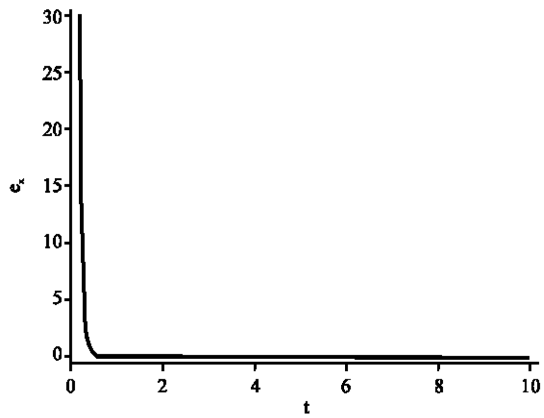

Fig. 4a: Shows the behaviour of the trajectory e_x of the error system tends to zero as t tends to 2 when the parameter values are a = 15, b = 5, c = 10 and r = 1

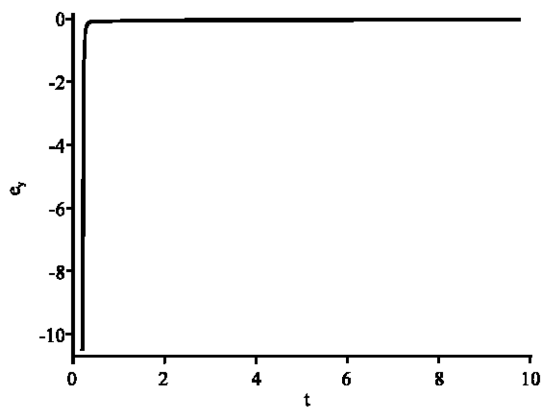

Fig. 4b: Shows the behaviour of the trajectory e_y of the error system tends to zero as t tends to 2 when the parameter values are a = 15, b = 5, c = 10 and r = 1

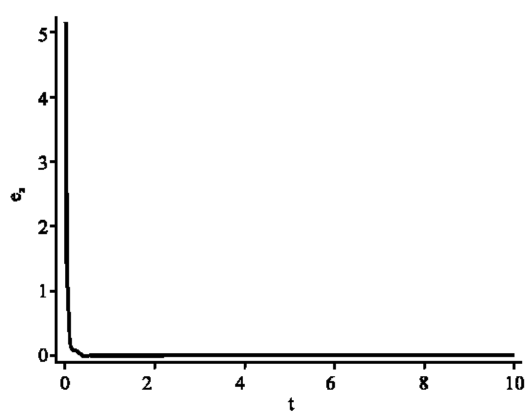

Fig. 4c: Shows the behaviour of the trajectory e_z of the error system tends to zero as t tends to 2 when the parameter values are a = 15, b = 5, c = 10 and r = 1

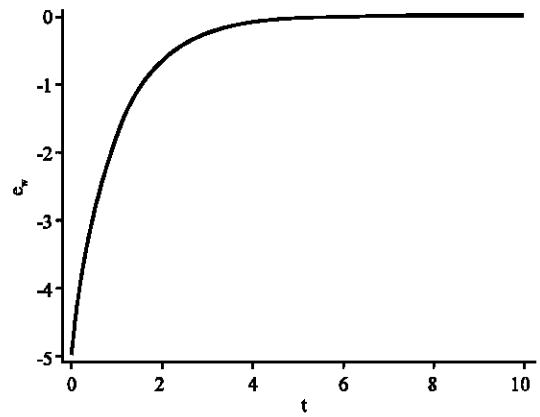

Fig. 4d: Shows the behaviour of the trajectory e_w of the error system tends to zero as t tends to 8 when the parameter values are a = 15, b = 5, c = 10 and r = 1

-20 < x < 20, -15 < y < 15, -20 < z < 20 and -20 < w < 20

Figure 4a-d shows that the trajectories of $e_x(t)$, $e_y(t)$, $e_z(t)$ and $e_w(t)$ of the error system tended to zero for $k_1 = 84$ and $k_2 = 1$.

ADAPTIVE SYNCHRONIZATION BETWEEN HYPERCHAOTIC LÜ SYSTEMS AND COUPLED VAN DER POL OSCILLATORS

In order to observe the adaptive synchronization behaviour in coupled van der Pol oscillators, we have the hyperchaotic Lü system is the drive system with four state variables denoted by the subscript 1 drives coupled van der Pol oscillators (response system) denoted by the subscript 2. Therefore hyperchaotic Lü system is described by the following Equations:

$$\begin{aligned}
\dot{x}_1 &= a(y_1 - x_1) \\
\dot{y}_1 &= -x_1 z_1 + c y_1 + w_1 + d_1(y_2 - y_1) \\
\dot{z}_1 &= x_1 y_1 - b z_1 \\
\dot{w}_1 &= z_1 - r w_1 + d_2(w_2 - w_1)
\end{aligned} \tag{15}$$

and coupled van der Pol oscillators is described by the following Equations:

$$\begin{aligned}
\dot{x}_2 &= y_2 + u_1 \\
\dot{y}_2 &= \varepsilon_1 (1 - x_2^2) y_2 - x_2 + \alpha(\varepsilon_2 w_2 + \omega_1^2 z_2 - \lambda x_2) + d_1(y_1 - y_2) + u_2 \\
\dot{z}_2 &= w_2 + u_3 \\
\dot{w}_2 &= -\varepsilon_2 w_2 - \omega_1^2 z_2 + \lambda x_2 + d_2(w_1 - w_2) + u_4
\end{aligned} \tag{16}$$

We have introduced four control inputs, u_1, u_2, u_3 and u_4 in Eq. 16, u_1, u_2, u_3 and u_4 are to be determined for the purpose of synchronizing the hyperchaotic Lü systems with the coupled van der Pol oscillators.

Let us define the state errors between the response system that is to be controlled and the controlling drive system as:

$$e_x = x_2 - x_1,\ e_y = y_2 - y_1,\ e_z = z_2 - z_1 \text{ and } e_w = w_2 - w_1$$

Then the error dynamical system can be written as

$$\begin{aligned}
\dot{e}_x &= y_2 + ax_1 - ay_1 + u_1 \\
\dot{e}_y &= \varepsilon_1 (1 - x_2^2) y_2 - x_2 + \alpha(\varepsilon_2 w_2 + \omega_1^2 z_2 - \lambda x_2) - 2d_1 e_y + x_1 z_1 - cy_1 - w_1 + u_2 \\
\dot{e}_z &= w_2 - x_1 y_1 + bz_1 + u_3 \\
\dot{e}_w &= -\varepsilon_2 w_2 - \omega_1^2 z_2 + \lambda x_2 - 2d_2 e_w - z_1 + rw_1 + u_4
\end{aligned} \tag{17}$$

suppose that

$$\dot{d}_1 = k_1 e_y^2 \text{ and } \dot{d}_2 = k_2 e_w^2 \text{ such that } k_1,\ k_2 > 0 \tag{18}$$

Then the synchronization problem is now replaced by the equivalent problem of stabilizing the system (17) using a suitable choice of the control laws u_1, u_2, u_3 and u_4.

Consider a Lyapunov function as follows:

$$V(e_x, e_y, e_z, e_w, d_1, d_2) = \frac{1}{2}(e_x^2 + e_y^2 + e_z^2 + e_w^2 + \frac{2}{k_1}(d_1 - d_1^*)^2 + + \frac{2}{k_2}(d_2 - d_2^*)^2) \tag{19}$$

where d_1^* and d_2^* are positive constants which will be defined later. Taking the time derivative of Eq. 19, then we get

$$\begin{aligned}
\dot{V} &= e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z + e_w\dot{e}_w + \frac{2}{k_1}(d_1 - d_1^*)\dot{d}_1 + + \frac{2}{k_2}(d_2 - d_2^*)\dot{d}_2 \\
&= e_x(y_2 + ax_1 - ay_1 + u_1) + e_z(w_2 - x_1 y_1 + bz_1 + u_3) \\
&\quad + e_y(\varepsilon_1 (1 - x_2^2) y_2 - x_2 + \alpha(\varepsilon_2 w_2 + \omega_1^2 z_2 - \lambda x_2) - 2d_1 e_y + x_1 z_1 - cy_1 - w_1 + u_2) \\
&\quad + e_w(-\varepsilon_2 w_2 - \omega_1^2 z_2 + \lambda x_2 - 2d_2 e_w - z_1 + rw_1 + u_4) + 2(d_1 - d_1^*)e_y^2 + 2(d_2 - d_2^*)e_w^2
\end{aligned}$$

There are many possible choices for the controller functions. We choose

$$\begin{aligned}
u_1 &= ay_1 - y_2 - ax_2 \\
u_2 &= x_2 + \varepsilon_1 x_2^2 y_2 - \alpha(\varepsilon_2 w_2 + \omega_1^2 z_2 - \lambda x_2) - x_1 z_1 + w_1 + (c - \varepsilon_1) y_1 \\
u_3 &= x_1 y_1 - w_2 - bz_2 \\
u_4 &= \varepsilon_2 w_1 + (1 + \omega_1^2) z_2 - \lambda x_2 - rw_2
\end{aligned} \tag{20}$$

under with this choice the error dynamical system is

$$\begin{aligned}
\dot{e}_x &= -ae_x \\
\dot{e}_y &= -(2d_1 - c - \varepsilon_1)e_y \\
\dot{e}_z &= -be_z \\
\dot{e}_w &= -(2d_2 + \varepsilon_2 + r)e_w + e_z
\end{aligned} \tag{21}$$

Therefore

$$
\begin{aligned}
\dot{V} &= -ae_x^2 - (2d_1^* - c - \varepsilon_1)e_y^2 - be_z^2 - (2d_2^* + r + \varepsilon_2)e_w^2 + e_z e_w \\
&= -[ae_x^2 + (2d_1^* - c - \varepsilon_1)e_y^2 + be_z^2 + (2d_2^* + r + \varepsilon_2)e_w^2 - e_z e_w] \\
&= -\begin{bmatrix}|e_x| & |e_y| & |e_z| & |e_w|\end{bmatrix}\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 2d_1^* - c - \varepsilon_1 & 0 & 0 \\ 0 & 0 & b & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 2d_2^* + r + \varepsilon_2 \end{bmatrix}\begin{bmatrix}|e_x| \\ |e_y| \\ |e_z| \\ |e_w|\end{bmatrix} \\
&= -\begin{bmatrix}|e_x| & |e_y| & |e_z| & |e_w|\end{bmatrix}\Psi(d_1^*, d_2^*)\begin{bmatrix}|e_x| & |e_y| & |e_z| & |e_w|\end{bmatrix}^T
\end{aligned} \tag{22}
$$

If $d_1^* > \frac{c+\varepsilon_1}{2}$ and $d_2^* > \frac{1}{8b} - \frac{r+\varepsilon_2}{2}$ then the 4×4 matrix $\Psi(d_1^*, d_2^*)$ is positive definite.

If d_1^* and d_2^* are appropriately chosen such that the 4×4 matrix $\Psi(d_1^*, d_2^*)$ in Eq. 22 is positive definite, then $\dot{V} \le 0$ holds. Since V is a positive and decreasing function and $\dot{V}$ is negative semidefinite (we chose $d_1^* > \frac{c+\varepsilon_1}{2}$ and $d_2^* > \frac{1}{8b} - \frac{r+\varepsilon_2}{2}$). It follows that the equilibrium point $(e_x = 0, e_y = 0, e_z = 0, e_w = 0, d_1 = d_1^*$ and $d_2 = d_2^*)$ of the system (21) is uniformly stable, i.e., $e_x(t), e_y(t), e_z(t), e_w(t) \in L_\infty$ and $d_1(t), d_2(t) \in L_\infty$. From Eq. 19 we can easily show that the squares of $e_x(t), e_y(t), e_z(t)$ and $e_w(t)$ are integrable with respect to time t, i.e., $e_x(t), e_y(t), e_z(t), e_w(t) \in L_2$. Next by Barbalat's Lemma Eq. 21 implies that $\dot{e}_x(t), \dot{e}_y(t), \dot{e}_z(t), \dot{e}_w(t) \in L_\infty$, which in turn implies $e_x(t) \to 0, e_y(t) \to 0, e_z(t) \to 0$ and $e_w(t) \to 0$ as $t \to \infty$. Thus, in the closed-loop system $x_2(t) \to x_1(t), y_2(t) \to y_1(t), z_2(t) \to z_1(t), w_2(t) \to w_1(t)$ as $t \to \infty$. This implies that the hyperchaotic Lü systems and coupled van der Pol oscillators have been globally asymptotically synchronized under the control law (20) associated with (18).

Numerical Experiment

By using the mathematical package Maple we solve the Eq. 15, 16 and 17. The four parameters are chosen as a = 15, b = 5, c = 10 and r = 1 in all simulations so that the hyperchaotic Lü system exhibits a chaotic behaviour if no control is applied. The four parameters are chosen as $\varepsilon_1 = 3.872$, $\varepsilon_2 = 0.000645$, $\lambda = 9.12$, $\alpha = 0.457$ and $\omega^2 = 5$ in all simulations so that the coupled van der

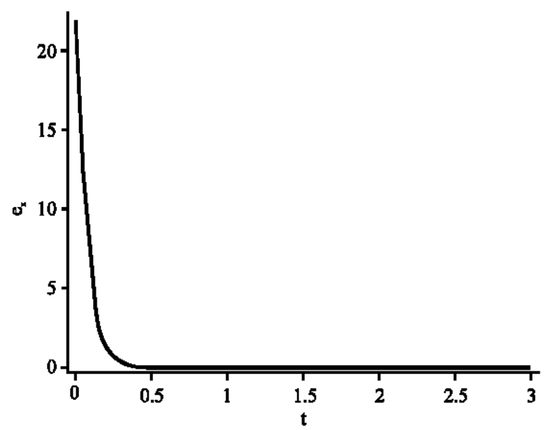

Fig. 5a: Shows the behaviour of the trajectory e_x of the error system tends to zero as t tends to 2 when the parameter values are a = 15, b = 5, c = 10, r = 1, $\varepsilon_1 = 3.872$, $\varepsilon_2 = 0.000645$, $\lambda = 9.12$, $\alpha = 0.457$ and $\omega^2 = 5$

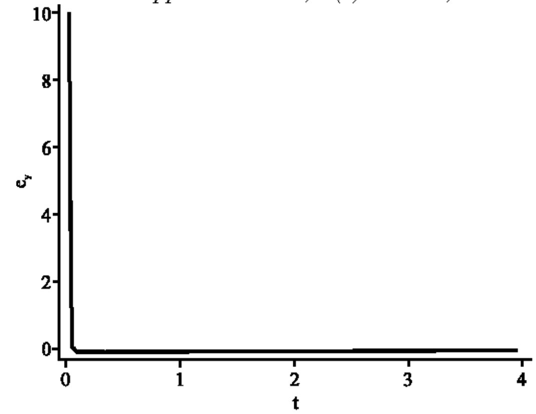

Fig. 5b: Shows the behaviour of the trajectory e_y of the error system tends to zero as t tends to 2 when the parameter values are a = 15, b = 5, c = 10, r = 1, $\varepsilon_1 = 3.872$, $\varepsilon_2 = 0.000645$, $\lambda = 9.12$, $\alpha = 0.457$ and $\omega^2 = 5$

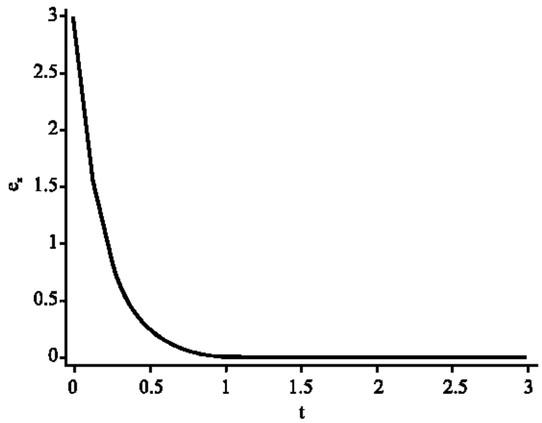

Fig. 5c: Shows the behaviour of the trajectory e_z of the error system tends to zero as t tends to 2 when the parameter values are a = 15, b = 5, c = 10, r = 1, $\varepsilon_1 = 3.872$, $\varepsilon_2 = 0.000645$, $\lambda = 9.12$, $\alpha = 0.457$ and $\omega^2 = 5$

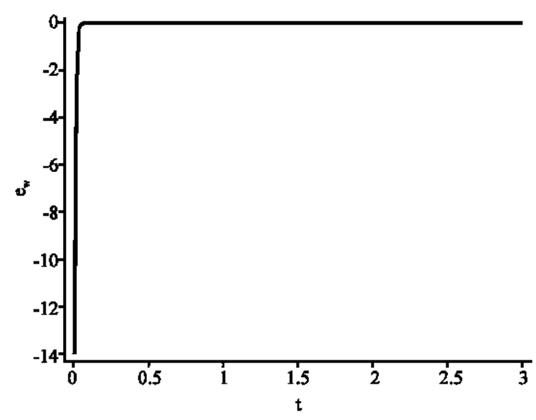

Fig. 5d: Shows the behaviour of the trajectory e_w of the error system tends to zero as t tends to 8 when the parameter values are a = 15, b = 5, c = 10, r = 1, $\varepsilon_1 = 3.872$, $\varepsilon_2 = 0.000645$, $\lambda = 9.12$, $\alpha = 0.457$ and $\omega^2 = 5$

Pol oscillators exhibits a chaotic behaviour if no control is applied. The initial states of the drive system are $x_1(0)=-20,\ y_1(0)=5,\ z_1(0)=0$ and $w_1(0)=15$ and of the response system are $x_2(0)=10,\ y_2(0)=-5,\ z_2(0)=5$ and $w_2(0)=10$. Then $e_x(0)=30,\ e_y(0)=-10,\ e_z(0)=5$ and $e_w(0)=-5$. In this case, we assume that the drive system is hyperchaotic Lü system and the response system is coupled van der Pol oscillators are different initial conditions. Figure 5a-d shows that the trajectories of $e_x(t),\ e_y(t),\ e_z(t)$ and $e_w(t)$ of the error system tended to zero for $k_1=10$ and $k_2=1$. These numerical results demonstrate the systems have been asymptotically synchronized using the proposed adaptive schemes.

CONCLUSIONS

In this study the adaptive synchronization problem of hyperchaotic Lü system and an electronic circuit consisting of a van der Pol oscillator coupled to a linear oscillator have been investigated. All results are proved by using Lyapunov direct method. The proposed scheme is efficient in achieving simple synchronization in our example and can be applied to similar chaotic systems.

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