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The Relations Among the Order Statistics of Uniform Distribution

[1]S.A. Moghadam and [2]H. Pazira
[1]Department of Statistics, Payame Nour University of Amol, Iran
[2]Department of Statistics, Payame Nour University of Tehran, Iran

Corresponding Author: Hassan Pazira, Department of Statistics, Payame Nour University of Tehran, Iran

ABSTRACT

There is a relation between variance and covariance of the order statistics of the continuous uniform distribution, so by using it, we can easily compute the moments of order statistics and specially the correlation of them. In this study, we present two theorems and prove them after expressing the general form of some basic moments. In the end, we present appendix consisted of the applied theorems.

Key words: Order statistics, moment, variance and covariance, continuous uniform distribution, order statistics transformation theorem, identically distributed theorem

INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ be a random sample from $U(\theta_1, \theta_2)$ distribution. We show the order statistic with $X_{(i)}$ for this sample. We know:

$$F_X(x) = \frac{x-\theta_1}{\theta_2-\theta_1}; \ \theta_1 \le x \le \theta_2$$

where, the parameter θ_1 and θ_2 satisfy in $-\infty<\theta_1<\theta_2<\infty$.

According to the order statistics transformation theorem (Appendix):

$$F(X_{(i)}) \overset{d}{=} V_i \sim \mathrm{Bet}(i, n+1-i)$$

thus:

$$\frac{X_{(i)}-\theta_1}{\theta_2-\theta_1} \overset{d}{=} V_i$$

where, according to identically distributed theorem (Appendix):

$$X_{(i)} \overset{d}{=} (\theta_2-\theta_1)V_i + \theta_1 \quad (1)$$

Now, since $X_{(i)}$ and $(\theta_2-\theta_1)\ V_i+\theta_1$ are identically distributed, they have a same density and moments and have a same moment generating function generally (if exist) (Behbodian, 2003b).

Therefore, when we intend to compute, for example, variance of $X_{(i)}$, we can use the variance of $(\theta_2-\theta_1)\ V_i+\theta_1$, because they have a equal result.

MOMENTS OF ORDER STATISTICS

Here, we present and prove some important moments of the order statistics of the random variable X ($X \sim U(\theta_1, \theta_2)$). For universalization this moment to all of the order statistics, this moments be computed generally. Also these moments are useful to proof of the theorem 1:

$$E(X_{(i)}) = \frac{i\theta_2 + (n+1-i)\theta_1}{n+1} \tag{a}$$

$$E(X_{(i)}X_{(j)}) = \frac{\left[i(j+1)\theta_2{}^2 + [(n+2)(j-i) + 2i(n+1-j)]\ \theta_1\theta_2 + (n+1-j)(n+2-i)\theta_1^2\right]}{(n+1)(n+2)} \tag{b}$$

where, $1 \le i < j \le n$, i, $j \in N$:

$$E(X_{(i)}^2) = \frac{\left[i(i+1)\theta_2{}^2 + 2i(n+1-i)\theta_1\theta_2 + (n+1-i)(n+2-i)\theta_1{}^2\right]}{(n+1)(n+2)} \tag{c}$$

$$V(X_{(i)}) = \frac{i(n+1-i)(\theta_2-\theta_1)^2}{(n+1)^2(n+2)} \tag{d}$$

$$Cov(X_{(i)}, X_{(j)}) = \frac{i(n+1-j)(\theta_2-\theta_1)^2}{(n+1)^2(n+2)} \tag{e}$$

where, $1 \le i < j \le n$, i, $j \in N$:

$$Corr(X_{(i)}, X_{(j)}) = \sqrt{\frac{i(n+1-j)}{j(n+1-i)}} \tag{f}$$

where, $1 \le i < j \le n$, I, $j \in N$.

According to the above equations, these equations are easily obtainable; as Eq. a, b and c are given by using the expectation definition, Eq. 1 and identically distributed theorem. Equation e and d are given by definition of variance and covariance and using a, b and c and last equation is given by using Eq. e and d. As an example, we obtain equation c.

We know, by using Eq. 1 and identically distributed theorem, it is sufficient to use the expectation of $[(\theta_2-\theta_1)\ V_i+\theta_1]^2$ in stead of the expectation of $X_{(i)}{}^2$, where $V_i \sim$ Bet(i,n+1−i). Thus we must compute $E(V_i)$ and $E(V_i^2)$:

$$E(V_i) = \frac{1}{B(i,n+1-i)}\int_0^1 v^i(1-v)^{n-i}\,dv = \frac{B(i+1,n+1-i)}{B(i,n+1-i)}$$
$$= \frac{\Gamma(i+1)\Gamma(n+1-i)}{\Gamma(n+2)} \cdot \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n+1-i)} = \frac{i}{n+1}$$

also:

$$E(V_i^2) = \frac{i(i+1)}{(n+1)(n+2)}$$

thus:

$$\begin{aligned} E(X_{(i)}^2) &= E\left[(\theta_2-\theta_1)V_i+\theta_1)\right]^2 = E[(\theta_2-\theta_1)^2V_i^2+2\theta_1(\theta_2-\theta_1)V_i+\theta_1^2] \\ &= (\theta_2-\theta_1)^2E(V_i^2)+2\theta_1(\theta_2-\theta_1)E(V_i)+\theta_1^2 = (\theta_2-\theta_1)^2\frac{i(i+1)}{(n+1)(n+2)}+2\theta_1(\theta_2-\theta_1)\frac{i}{n+1}+\theta_1^2 \\ &= \frac{\left[i(i+1)\theta_2^2+2i(n+1-i)\theta_1\theta_2+(n+1-i)(n+2-i)\theta_1^2\right]}{(n+1)(n+2)} \end{aligned}$$

thus part (c) is true, (Behbodian, 2003a; Mood *et al.*, 1998).

Notice, since $0<Corr(X_{(i)}X_{(j)})\le 1$, order statistics of uniform distribution have a positive correlation.

We express the important result of above equations in the follow theorem.

Theorem 1: Let $X_1, X_2, \ldots, X_n$ be a random sample from $U(\theta_1, \theta_2)$ distribution with order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, then the follow equation always is true:

$$V(X_{(i)}) = V(X_{(n+1-i)}) = \begin{cases} \frac{n+1-i}{i}Cov(X_{(i)}, X_{(n+1-i)}); i<\frac{n+1}{2} \\ \frac{i}{n+1-i}Cov(X_{(i)}, X_{(n+1-i)}); i>\frac{n+1}{2} \end{cases} \quad (2)$$

where, $1\le i<j\le n$, $i, j\in N$.

Proof: We use the above moment equations for prove this theory. We use the part d for prove first equal in left hand:

$$\begin{aligned} V(X_{(n+1-i)}) &= \frac{(n+1-i)(n+1-(n+1-i))(\theta_2-\theta_1)^2}{(n+1)^2(n+2)} \\ &= \frac{i(n+1-i)(\theta_2-\theta_1)^2}{(n+1)^2(n+2)} = V(X_{(i)}) \end{aligned}$$

Thus the first equal is true. We use the part e for prove second equal.

First we consider $i<\frac{n+1}{2}$. Since $i<\frac{n+1}{2}$ that is $i<n+1-i$, we have:

$$Cov(X_{(i)}, X_{(n+1-i)}) = \frac{i(n+1-(n+1-i))(\theta_2-\theta_1)^2}{(n+1)^2(n+2)} = \frac{i^2(\theta_2-\theta_1)^2}{(n+1)^2(n+2)}$$

thus:

$$\frac{n+1-i}{i}Cov(X_{(i)}, X_{(n+1-i)}) = \frac{i(n+1-i)(\theta_2-\theta_1)^2}{(n+1)^2(n+2)} = V(X_{(i)})$$

Now, if $i>\frac{n+1}{2}$ that is i<n+1-i, we can write:

$$Cov(X_{(i)}, X_{(n+1-i)}) = \frac{(n+1-i)(n+1-i)(\theta_2-\theta_1)^2}{(n+1)^2(n+2)} = \frac{(n+1-i)^2(\theta_2-\theta_1)^2}{(n+1)^2(n+2)}$$

hence:

$$\frac{i}{n+1-i} Cov(X_{(i)}, X_{(n+1-i)}) = \frac{i(n+1-i)(\theta_2-\theta_1)^2}{(n+1)^2(n+2)} = V(X_{(i)})$$

and proof is complete.

We know that, if we sort $X_1,X_2,....,X_n$ as ascendant (so that obtained the order statistics), $X_{(i)}$ and $X_{(n+1-i)}$ are the order statistics, so that, with respect to the median $X_{(\frac{n+1}{2})}$ or:

$$\frac{X_{(\frac{n}{2})} + X_{(\frac{n}{2}+1)}}{2}$$

are concurrent. Indeed, those have equal distance to median and theorem 1 use for such statistics.

Theorem 2: Let $X_1,X_2,....,X_n$ be a random sample from $U(\theta_1, \theta_2)$ distribution with order statistics $X_{(1)},X_{(2)},....,X_{(n)}$. Then both following formula are true if $1 \le i<j \le n$:

$$X_{(j)} - X_{(i)} \stackrel{d}{=} X_{(j-i)} - \theta_1 \tag{3}$$

$$X_{(i)} + X_{(j)} \stackrel{d}{=} X_{(i+j)} + \theta_1 \tag{4}$$

Proof: For prove, we define the order statistics $X_{(i)}$ and $X_{(j)}$ with attention to Eq. 1 and condition $1 \le i<j \le n$:

$$X_{(i)} \stackrel{d}{=} (\theta_2-\theta_1)V_i + \theta_1$$

$$X_{(j)} \stackrel{d}{=} (\theta_2-\theta_1)V_j + \theta_1$$

According to the Identically distributed and order statistics transformation theorem (Appendix) we have:

$$X_{(j)} - X_{(i)} \stackrel{d}{=} (\theta_2-\theta_1)(V_j - V_i)$$

and since know (Behbodian, 2003b):

$$V_j - V_i \stackrel{d}{=} V_{(j-i)}$$

then by using Eq. 1 we can write:

$$X_{(j)} - X_{(i)} \stackrel{d}{=} (\theta_2 - \theta_1)V_{(j-i)} \stackrel{d}{=} X_{(j-i)} - \theta_1$$

so Eq. 3 is proved. With a little change in this proof, we can prove Eq. 4. According to identically distributed theorem we have:

$$X_{(i)} + X_{(j)} \stackrel{d}{=} (\theta_2 - \theta_1)(V_i + V_j) + 2\theta_1$$

and since we know (Behbodian, 2003b):

$$V_i + V_j \stackrel{d}{=} V_{(i+j)}$$

thus:

$$(\theta_2 - \theta_1)(V_i + V_j) + 2\theta_1 \stackrel{d}{=} (\theta_2 - \theta_1)V_{(i+j)} + 2\theta_1$$

by using Eq. 1 we can write:

$$X_{(i)} + X_{(j)} \stackrel{d}{=} (\theta_2 - \theta_1)V_{(i+j)} + 2\theta_1 \stackrel{d}{=} X_{(i+j)} + \theta_1$$

thus Eq. 4 is proved and the proof is complete.

APPENDIX

Identically distributed theorem: Let X and Y be random variables. If $X \stackrel{d}{=} Y$ (or $P(X \le z) = P(Y \le z) = F(z)$) and g(t) be a arbitrary function, then $g(X) \stackrel{d}{=} g(Y)$. (Behbodian, 2003b, Chapter 1, Theorem 2).

Order statistics transformation theorem: Let $X_1, X_2, ..., X_n$ be a random sample so that has a probability distribution function (p.d.f.) of the form f and a cumulative distribution function (p.d.f.) of the form F. We show the order statistics with $X_{(1)}, X_{(2)}, ..., X_{(n)}$. Let $U_1, U_2, ..., U_n$ be a random sample from U(0,1) distribution, we show the order statistics it with V_1, $V_2, ..., V_n$. Then we have (Behbodian, 2003b):

- $(F(X_1), ..., F(X_n)) \stackrel{d}{=} (U_1, U_2, ..., U_n)$
- $(F(X_{(1)}), ..., F(X_{(n)})) \stackrel{d}{=} (V_1, V_2, ..., V_n)$
- $F(X_{(k)}) - F(X_{(j)}) \stackrel{d}{=} V_k - V_j \stackrel{d}{=} V_{k-j}$
- $V_r \sim Bet(r, n-r+1)$

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