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The Existence of Weak Solution for a Calss of Nonlinear P(x)-boundry Value Problem Involving the Principle Eigenvalue

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ABSTRACT

The set of eigenvalue for the p (x)-laplacian Dirichlet problem is a nonempty set. Unfortunately in general case, the principle eigenvalue λ_* of this set is equal to zero, whereas for p (x)≡constant, the fact $\lambda_*>0$ is very important in the study of p-laplacian problems. In this study, we suppose some sufficient conditions to use of nonzero principle eigenvalue in variable exponent case to find the solutions for p (x)-boundary value problem:

$$\begin{cases} -\Delta_{p(x)} u(x) + \lambda u(x)^{p(x)-2} u(x) = g(x, t); \text{in } \Omega \\ \qquad u \equiv 0; \text{on } \partial\Omega \end{cases}$$

on a bounded subset of $\mathbb{R}^N$. which can be regarded as a starting point for investigating of models like those described in p-laplacian in which the principle eigenvalue is involved.

Key words: p(.)¯laplacian, variable exponent sobolev space, eigenvalue, critical point, weak solution

INTRODUCTION

Many problems in physics and mechanics can be modeled with sufficient accuracy using classical Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, where p is fixed constant and Ω is a appropriate domain. But for the electrorheological fluids (smart fluids) this is not adequate but rather, the exponent should be able to vary. This leads us to study of variable exponent Lebesgue and Sobolev spaces, $L^{p(.)}(\Omega)$ and $W^{1,p(.)}(\Omega)$, where is real-valued function.

An interesting mathematical model for electrorheological fluids is developed by Rajagopal and Ruzika. The model takes into account the delicate interaction between the electromagnetic field and the moving fluid. Particularly in the context of continuum mechanics, these fluids are seen as non-Newtonian fluid. This study can be regarded as an investigation of Mihailescu and Radulescu (2006).

Variational problems with the nonstandard growth condition has been studied extensively during the past decades and many interesting results have been obtained; for example see Agarwal *et al.* (2011), Alves and Souto (2005), Fan (2005), Fan *et al.* (2005), Ghaemi and Saiedinezhad (2011), Mihailescu and Radulescu (2006) and Samko (2005).

In present study, we consider for the problem (P):

$$(\mathrm{P})\begin{cases}-\Delta_{p(x)}u(x)+\lambda u(x)^{p(x)-2}u(x)=g(x,t); & \text{in }\Omega\\ u\equiv 0; & \text{on }\partial\Omega\end{cases}$$

which is similar to the constant exponent case. However, the variable exponent cases posses more complicated nonlinearities and this makes the problem difficult in this case. Finally, we derive existence result by applying the mountain pass method to problem (P).

PRELIMINARY

For basic definitions about variable exponent Lebesgue and Sobolev space we refer to Diening *et al.* (2004), Hudzik (1977) and Ruzicka (2000). Here, we mention some of main them.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p\epsilon(\bar{\Omega})$ and

$$1<p^-:=\text{ess}\inf_{x\in\Omega}p(x)<p^+:=\text{ess}\sup_{x\in\Omega}p(x)<\infty \tag{1}$$

The variable exponent Lebesgue space $L^{p(.)}(\Omega)$ is defined by:

$$L^{p(.)}(\Omega)=\left\{u;u;\Omega\to\mathbb{R}\text{ is masurable},\int_\Omega|u(x)|^{p(x)}dx<\infty\right\}$$

which is a Banach space with the norm:

$$u|_{L^{p(.)}(\Omega)}=\inf\{\sigma>0;\int_\Omega\left|\frac{u}{\sigma}\right|^{p(x)}dx\le 1\}$$

The variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ is defined by:

$$W^{1,p(.)}(\Omega)=\{u\in L^{p(.)}(\Omega);|\nabla u|\in L^{p(.)}(\Omega)\}$$

with the norm:

$$\|u\|_{W^{1,p(.)}(\Omega)}=|u|_{L^{p(.)}(\Omega)}+|\nabla u|_{L^{p(.)}(\Omega)}$$

Define $W_0^{1,p(.)}(\Omega)$ as the closure of C_0^∞ in $W^{1,p(.)}(\Omega)$ and

$$p^*(x)=\begin{cases}\dfrac{Np(x)}{N-p(x)}; & p(x)<N;\\ \infty; & p(x)\ge N\end{cases}$$

Then we have:

Proposition 1: (Fan and Zhao, 1998):

(1) $W^{1,p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$ are separable, reflexive Banach spaces
(2) If $q\epsilon C(\bar{\Omega})$ and $1\le q(x)<p^*(x)$ for all $x\epsilon\bar{\Omega}$ then there is a compact and continuous embedding from $W^{1,p(.)}(\Omega)$ to $L^{q(.)}(\Omega)$; it is shown with:

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$$

(3) There is a constant C>0, such that;

$$|u|_{L^{p(\cdot)}(\Omega)} \leq C|\nabla u|_{L^{p(\cdot)}(\Omega)};\forall u \in W_0^{1,p(\cdot)}(\Omega)$$

By (3) of proposition 1 we know that $|\nabla u|_{L^{p(\cdot)}(\Omega)}$ and $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$. We use $|\nabla u|_{L^{p(\cdot)}(\Omega)}$ to replace $\|u\| = \|u\|_{W^{1,p(\cdot)}(\Omega)}$ in the following discussion.

Proposition 2: (Zhao and Fan, 1998): If $f{:}\Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory function and satisfies:

$$|f(x,s)| \leq a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}};\forall x \in \Omega, s \in \mathbb{R}$$

Where, p_2, $p_2 \in C(\Omega)$ and $p_1(x)$, $p_2(x)>1$ for all $x \in \bar{\Omega}$.

Moreover, $a \in L^{p_2(\cdot)}(\Omega)$, $a(x)>0$ and $b\geq 0$ is constant, then the Nemytsky operator from $L^{p_1(\cdot)}(\Omega)$ to $L^{p_2(\cdot)}(\Omega)$ defined by $(N_f(u))(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 3: (Fan and Zhao, 2001): If we define ρ: $L^{p(x)}(\Omega)\to\mathbb{R}$ by:

$$\rho(x)=\int_\Omega |u(x)|^{p(x)}dx \, \forall u \in L^{p(x)}(\Omega)$$

Then;

$$\min\{|u|^{p+}_{L^{p(\cdot)}(\Omega)}, |u|^{p-}_{L^{p(\cdot)}(\Omega)}\} \leq \rho(u) \leq \max\{|u|^{p+}_{L^{p(\cdot)}(\Omega)}, |u|^{p-}_{L^{p(\cdot)}(\Omega)}\}$$

Consider the eigenvalue problem:

$$(P_0)\begin{cases} -\Delta_{p(x)}u(x)=\lambda u(x)^{p(x)-2}u(x); \text{ in } \Omega \\ u\equiv 0; \text{ on } \partial\Omega \end{cases}$$

Definition 1: Let $\lambda\in\mathbb{R}$ and $u\in W_0^{1,p(\cdot)}$. (u, λ) is called a pair of solution of problem (P_0) if:

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \lambda\int_\Omega |u|^{p(x)-2}uv dx;\ \forall v \in W_0^{1,p(\cdot)}(\Omega)$$

If (u, λ) is a solution of (P_0) and $u\neq 0$, we call λ and u eigenvalue and eigenfunction corresponding to λ of (P_0), respectively.

It is easy to see that, if (u, λ) is a solution of (P_0) and $u\neq 0$, then:

$$\lambda = \lambda(u) = \frac{\int_\Omega |\nabla u|^{p(x)}dx}{\int_\Omega |u|^{p(x)}dx}$$

and hence, $\lambda>0$. Define:

$$\lambda_* = \lambda_{*,p(\cdot)} := \inf\left\{\frac{\int_\Omega |\nabla u|^{p(x)}\,dx}{\int_\Omega |u|^{p(x)}\,dx};\ 0 \neq u \in W_0^{1,p(\cdot)}(\Omega)\right\} \tag{2}$$

Theorem 1: (Fan *et al.*, 2005): Let N>1. If there is a vector $l\in\mathbb{R}^N\setminus\{0\}$ such that for any $x\in\Omega$, $f(t) = p(x+tl)$ is monotone for $t\in I_x := \{t \mid x+tl\in\Omega\}$, then $\lambda_*>0$. If N = 1 then $\lambda_*>0$ if and only if p (x) is monotone.

Definition 2: Let $F\in C(X, \mathbb{R})$ where X is Banach space. For $c\in\mathbb{R}$ the functional F satisfies the Palais-Smale condition on the level c (shortly $(PS)_c$) if any sequence $\{x_n\}^\infty_{\{n=1\}}\subset X$ such that:

$$F(x_n)\to c,\ \nabla F(x_n)\to 0$$

has convergent subsequence (in the norm of X).

Theorem 2: (Drabek and Milota, 2007) (Mountain Pass Theorem): Let X be a Banach space and let $F\in C(X, \mathbb{R})$, $e\in X$ and $r>0$ be such that $|e|>r$ and

$$\inf_{\{x\in X, |x| = r\}} F(x) > F(0) \geq F(e)$$

If F satisfies the $(PS)_c$ condition then c is critical value of F.

EXISTENCE OF SOLUTIONS

Theorem 3: Let $p: \Omega\to\mathbb{R}$ be continuous function which satisfy (1) and assumptions of Theorem 1. $g: \Omega\times\mathbb{R}$ is a caratheodory function and satisfies:

$$|g(x, y)| \leq d_1 + d_2 |y|^{\alpha-1};\ \forall x\in\Omega,\ y\in\mathbb{R} \tag{3}$$

where, $p^+<\alpha<p^*(x)$ and d_1, d_2 are positive constants. Moreover:

$$\lim_{y\to 0}\frac{g(x,y)}{|y|\,p^{+-1}} = 0 \qquad \text{uniformly for all } x\in\Omega \tag{4}$$

and there exist $\vartheta > p^+$ such that:

$$g(x, y_1) \leq \frac{1}{\vartheta} g(x, y_2); \qquad \forall y_1 < y_2 \tag{5}$$

Then problem (p) has a weak solution provided that $\lambda>-\lambda_*$ where, λ_* is introduced in (2).

Proof: Let us define $F: W_0^{1,p(\cdot)}(\Omega)\to\mathbb{R}$ by:

$$F(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{\lambda}{p(x)}|u|^{p(x)}\,dx - \int_\Omega G(x,u(x))\,dx \tag{6}$$

Where;

$$G(x,s) = \int_0^s g(x,y)\,dy$$

Hence, $F \in C(X, \mathbb{R})$ and its critical points correspond to the weak solutions of (p) (Chang, 1986). Indeed:

$$(\nabla f, y) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_\Omega |\nabla u|^{p(x)-2} uvdx - g(x, u(x))v(x)dx; \qquad \forall v \in X$$

Note that: $\lambda_* > 0$ (Theorem 1) and so there exist $c_0 > 0$ such that for every $u \in X$ we obtain:

$$c_0 \int_\Omega |u|^{p(X)} dx \le \int_\Omega |\nabla u|^{p(X)} dx \tag{7}$$

for $\lambda > -\lambda_*$ the expression:

$$I_\lambda = \int_\Omega |\nabla u|^{p(X)} dx + \lambda \int_\Omega |u|^{p(X)} dx$$

Satisfies:

$$I_\lambda(u) \ge c_1 \rho(\nabla u); \qquad \text{for any } u \in X \tag{8}$$

where, $c_1 = c_2(\lambda) = 1 + \min\{0, \lambda/\lambda_*\}$, is constants independent of u. Hence, by applying proposition 3 we obtain:

$$\begin{aligned} &\text{i. } |u| \ge 1 \Rightarrow I_\lambda(u) \ge c_1 \|u\|^{p-} \\ &\text{ii. } |u| \ge 1 \Rightarrow I_\lambda(u) \ge c_2 \|u\|^{p+} \end{aligned} \tag{9}$$

Now from (3) and (4) we have for every $\varepsilon > 0$:

$$G(x, s) \le \varepsilon |s|^{p+} + C(\varepsilon)\ |s|^\alpha$$

Since $p^+ < p^*(x)$, we have $X \to L^{p^*}(\Omega)$ and so there exist $\xi > 0$ such that:

$$\|u\|_{L^{p^+}(\Omega)} \le \xi \|u\|; \forall u \in X$$

Now let $\varepsilon > 0$ be small enough such that:

$$\varepsilon \xi^{p+} \le \frac{c_2}{2^{p+}}$$

Hence, for $\|u\| < 1$ we have:

$$
\begin{aligned}
F(u) &\geq \frac{1}{p^+} I_\lambda(u) - \varepsilon\int_\Omega |u|\, p^+\, dx - C(\varepsilon)\int_\Omega |u|^\alpha\, dx \\
&\geq \frac{1}{p^+} c_1 \|u\|^{p+} - \varepsilon\xi^{p+} - C(\varepsilon)\|u\|^\alpha \\
&\geq \frac{c_1}{2p^+}\|u\|^{p+} - C(\varepsilon)\|u\|^\alpha
\end{aligned}
$$

Therefore, there exist r>0 (small enough) such that $b = \inf_{\|u\|=r} F(u) > 0 = F(0)$.

Now let $u \in X \setminus \{0\}$ and $\|u\|<1$ then for $\kappa>1$:

$$F(\kappa u) \leq \frac{\kappa^{p+}}{p^-} I_\lambda(u) - \int_a^b G(x, \kappa u(x))\, dx$$

By assumption (5) For some good constants C, M>0 we have:

$$G(x,s) = \int_0^s g(x,y)\, dy \geq \int_0^s \frac{\vartheta}{y} G(x,y)\, dy \geq C\int_0^s \frac{\vartheta}{y}\, dy \geq C|s|^\vartheta; \qquad |s| > M$$

Recall (7) then we obtain:

$$I_\lambda(u) \leq \left(1 \frac{\lambda}{c_0}\right)\rho(\nabla u)$$

Hence:

$$F(\kappa u) \leq \frac{\kappa^{p+}}{p^-}\left(1\frac{\lambda}{c_0}\right)\|u\|\, p^- - C\kappa^\vartheta \int_0^s |u|\, dx$$

which implies for large κ we have $F(\kappa u) \leq 0$.

Set $e = \kappa u$ then we obtain $\|e\|>r$ and $F(e) \leq 0$. It remains to verify that F satisfies the $(ps)_c$ condition. Actually we will verify that F satisfies even a stronger version of $(ps)_c$ condition. Namely we prove that any sequence $\{u_n\}_1^\infty$ satisfying $d := \sup_n F(u_n) < \infty$ and $F(u_n) \to 0$ contain a convergent subsequence.

By using (5), we obtain:

$$\int_\Omega (G(x, u_n(x))\, dx \leq \frac{1}{\vartheta}\int_\Omega g(x, u_n(x)\, dx$$

and hence, for $\|u_n\|>1$.

$$
\begin{aligned}
d \geq F(u_n) &\geq \frac{1}{p^+} I_\lambda(u_n) - \int_a^b G(t, u_n)\, dx > \frac{1}{p^+} I_\lambda(u_n) - \frac{1}{\vartheta}\int_\Omega g(x, u_n) u_n\, dx \\
&= \leq \left(\frac{1}{p^+} - \frac{1}{\vartheta}\right) I_\lambda(u_n) + \frac{1}{\vartheta}(F'(u_n), u_n) \geq \left(\frac{1}{p^+} - \frac{1}{\vartheta}\right) c_1 \|u_n\|\, p^- - \frac{1}{\vartheta}\|F'(u_n)\|\|u_n\|
\end{aligned}
$$

We know $p^+ < \vartheta$ and so $\|u_n\|$ is bounded. Now, passing to a subsequence if necessary, we can assume that $u_n \to u$ in $W_0^{1,p(.)}(\Omega)$. By the compact embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\Omega)$, we have $u_n \to u$ in $L^{p(.)}(\Omega)$.

On the other hand:

$$(F'(u_n) - F'(u), u_n - u) = I_\lambda(u_n - u) - \int_\Omega g(x, u_n)(u_n - u)\,dx$$

It is clear that $(F'(u_n)-F'(u), u_n-u) \to 0$ as $n \to \infty$. The uniform converges of $\{u_n\}_1^\infty$ and $\{g(., u_n(.)\}_1^\infty$ implies that also:

$$\int_\Omega (g(x, u_n) - g(x, u_n))(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty$$

Therefore:

$$I_\lambda(u_n - u) \to 0 \qquad \text{as } n \to \infty$$

and by use of (9), $u_n \to u$ in $W_0^{1,p(.)}(\Omega)$.

It follows from theorem 2 that there exist a critical point $u_0 \epsilon X$ of F and hence a weak solution of (p).

Corollary 1: Suppose the conditions in theorem 3 are hold, moreover g (x, s) = 0 when s<0, then problem (p) has a non negative weak solution.

Proof: It follows from Theorem 3 that there exist weak solution $u_0 \epsilon$of $W_0^{1,p(.)}(\Omega)$ and so:

$$\int_\Omega |\nabla u_0|^{p(x)-2} \nabla v dx + \int_\Omega |u_0|^{p(x)-2} u_0 v dx - \int_\Omega g(x, u_0(x)) v(x) dx; \qquad \forall v | \in W_0^{1,p(.)}(\Omega)$$

Taking $v_0^- = \max\{o, u_0\}$, we get:

$$\int_\Omega \left|\frac{dv_0^-}{dx}\right|^{p(x)} dx + \lambda \int_\Omega |v_0^-|^{p(x)} dx = 0$$

Hence, $I_\lambda(v_0^-) = 0$ and $\|v_0^-\| = 0$, i.e., $v_0^- \geq 0$ for all $x \epsilon \Omega$.

Remark 1: Theorem 3 is valid, if we replace the number α with the $\alpha(x) \epsilon C(\Omega)$ where $\alpha(x) > p(x)$.

Theorem 4: (Drabek and Milota, 2007): Suppose $u_0 \epsilon C^1(\Omega)$ be local extermum of F, where $F(u) = f(x, u(x), \nabla u(x))\,dx$ and $f = f(x, r, s)$ is a function defined on $\Omega \times \mathbb{R}^2$ with continuous second partial derivatives respect to all its variable and let $x_0 \epsilon \Omega$ be such that:

$$\left|\frac{\partial^2 f}{\partial s^2}\right|(x, u_0(x_0), \nabla u_0(x_0)) \neq 0$$

Then there exist $\delta > 0$ such that $u_0 \epsilon C^2(x_0 - \delta, x_0 + \delta)$.

Corollary 2: Suppose the conditions in theorem 3 are hold, moreover

$$\frac{\partial g}{\partial x}, \frac{\partial g}{\partial s}$$

are continuous functions. If the problem (p) has the non constant weak solution $u_0 \in W_0^{1,p(.)}(\Omega)$ then there exists $x_0 \in \Omega$ and $\delta > 0$ such that $u_0 \in C^2(x_0-\delta, x_0+\delta)$.

Proof: Apply theorem 4 with:

$$f(x,r,s) = \frac{1}{p(x)}|s|^{p(x)} + \frac{\lambda}{p(x)}|r|^{p(x)} - G(x,r).$$

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