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On Stability Analysis of Nonlinear Systems

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ABSTRACT

In this study, the behavior of a second-order dynamical system around its equilibrium point was analyzed based on the behavior of some appropriate equipotential curves which were considered around the same equilibrium point. In fact two sets of equipotential curves were considered so that a set of the equipotential curves had a role as the upper band of the system trajectory and another set played a role as the lower band. It was shown that stability of the system around its equilibrium point can be assessed using the behavior of these two set of equipotential curves. It was shown that asymptotically stability and instability analysis of the system only need the analysis of the upper band set of the equipotential curves but oscillation behavior analysis of the system need to analyze both the lower band set of the equipotential curves and the upper band set. Also as was shown the proposed method is a mathematical method that can even detect a stable limit cycle appearing in the oscillation systems and furthermore the method has many applications such as designing of oscillators that were presented.

Key words: Equilibrium point, stability, instability, autonomous system

INTRODUCTION

The stability analysis of a nonlinear system is very important and difficult problem. In fact, there are not any assumptions to start the stability analysis of a nonlinear system. A very simple nonlinear system can be unstable while a very complex nonlinear system can be stable. When a system is designed, the first important problem is to guarantee the system stability. The nonlinearity of a system results many various behaviors that makes impossibility to classify the nonlinear systems in distinguished categories and this makes very difficult to analyze the stability of a nonlinear system. For this reason, researches on the stability analysis of a nonlinear system have been always interesting. To days, new methods for stability analysis of a nonlinear system are very appealing and necessary especially for system designers (Vidyasagar, 1993; Thomsen, 2003).

There are two classic methods which are essentially used to analyze the stability of a nonlinear system (Vidyasagar, 1993; Doyle *et al.*, 1992). The first method is stability analysis using energy function and the second method is based on the linearization the system around its equilibrium point. Sometimes the first method presented by Lyapunov is called "Direct Method" and the second method is called "Indirect Method". The complexity of the first method is to find the appropriate energy functions to assess the stability of a nonlinear system. The second method has two major weaknesses. Firstly, if the eigenvalues of the coefficient matrix in the linear system resulted of linearization have real parts that are equal zero, the original nonlinear system may be stable or instable around the equilibrium point. Secondly, the method can only assess stability as the form of local stability around the equilibrium point (Vidyasagar, 1993).

Also two basic methods are essentially used to analyze the behavior of limit cycles appearing in nonlinear system (Vidyasagar, 1993; Doyle *et al.*, 1992). The first method is to draw the trajectories of the system using the softwares such as MATLAB in order to detect the limit cycles of the system. It is clear that the limit cycles detected by this approach can be recognized as stable, unstable or semi-stable (Vidyasagar, 1993; Doyle *et al.*, 1992). The second method is based on the linearization of the system around its equilibrium point or points. The second method has two major weaknesses. Firstly, the equilibrium point of the system, which the system is linearized around it, must be on the limit cycle or in a small neighborhood of it otherwise the method can not detect the real behavior of the nonlinear system. Secondly, the method can only assess the behavior of the system around the limit cycle as the form of point to point if and only if these points all locate on the limit cycle (Vidyasagar, 1993). There are some new researches done in recent years that present some mathematical methods to analyze the stability and behavior of some special nonlinear systems (Nan and Weber, 2010; Mann and Shiller, 2006; Benkler *et al.*, 2006; Marro and Zattoni, 2002; Ntogramatzidis *et al.*, 2007; Martinez *et al.*, 2003; Sheheitli and Rand, 2011; Thomsen, 2003). These methods can not be applied for analyzing of general second or higher order nonlinear systems.

The proposed method mentioned in this paper is a mathematical method which is suitable to analyze the stability of second-order nonlinear autonomous systems. These kinds of systems are very important because they model the behavior of some devices such as oscillators (Doyle *et al.*, 1992). The proposed method does not have any limitations and it can even detect a stable limit cycle appearing in the oscillation systems and furthermore the method has many applications such as designing of oscillators.

EQUIPOTENTIAL CURVES

Consider the second-order autonomous system described by the following equation:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \tag{1}$$

Definition 1: A set is said "compact" if it is bounded and closed (Vidyasagar, 1993).

Definition 2: Consider the set called P so that $P \subset R^2$, P is said "invariant set" if the trajectories of the system beginning in the P remain in it as $t \to \infty$ (Vidyasagar, 1993).

Definition 3: Suppose that the X = 0 is the equilibrium point of the second-order autonomous system described by the Eq. 1 and suppose that the compact set called M includes the equilibrium point (the origin). The closed curves belonging the M, which is described by $u(x_1, x_2) = C$ so that $C \in R$ and enclosing the equilibrium point, are called equipotential curves because for each value of C there is a closed curve with the potential of C, so all points locating on the $u(x_1, x_2) = C$ have the equal potential the numerical quantity of which is C.

STABILITY ANALYSIS

Theorem 1: The second-order autonomous system described by Eq. 1 is local asymptotic stable around the equilibrium point (X = 0) if there are equipotential curves $u(x_1, x_2) = C$ with clockwise direction, enclosing the equilibrium point and further on the trajectories of the system 1:

$$\frac{du(x_1, x_2)}{dt} < 0 \tag{2}$$

Proof: From $u(x_1, x_2) = C$, we have:

$$\frac{\partial u(x_1, x_2)}{\partial x_1}\dot{x}_1^* + \frac{\partial u(x_1, x_2)}{\partial x_2}\dot{x}_2^* = 0 \tag{3}$$

and as a result, the dynamic of $u(x_1, x_2) = C$ can be expressed as:

$$\begin{cases} \dot{x}_1^* = \dfrac{\partial u(x_1, x_2)}{\partial x_2} \\ \dot{x}_2^* = -\dfrac{\partial u(x_1, x_2)}{\partial x_1} \end{cases} \tag{4}$$

where, $\dot{x}_1^*$ and $\dot{x}_2^*$ are the state variables of the dynamic of $u(x_1, x_2) = C$. The velocity vector on the $u(x_1, x_2) = C$ symbolized by $\vec{V}_u$ is define as $\vec{V}_u = \dot{x}_1^*\vec{u}_{x_1} + \dot{x}_2^*\vec{u}_{x_2}$, so from Eq. 4 we found that:

$$\vec{V}_u = \frac{\partial u(x_1, x_2)}{\partial x_2}\vec{u}_{x_1} + \left(-\frac{\partial u(x_1, x_2)}{\partial x_1}\right)\vec{u}_{x_2} \tag{5}$$

where, $\vec{u}_{x_1}$ and $\vec{u}_{x_2}$ are respectively the unity vectors of the x_1 axis and x_2 axis. Also, the velocity vector of the system 1 is defined as $\vec{\dot{X}} = \dot{x}_1\vec{u}_{x_1} + \dot{x}_2\vec{u}_{x_2}$, so it can be written as:

$$\vec{\dot{X}} = f_1(x_1, x_2)\,\vec{u}_{x_1} + f_2(x_1, x_2)\,\vec{u}_{x_2} \tag{6}$$

The derivative:

$$\frac{du(x_1, x_2)}{dt}$$

on the trajectories of the system 1 can be expressed as:

$$\frac{du(x_1, x_2)}{dt} = \frac{\partial u(x_1, x_2)}{\partial x_1}\dot{x}_1 + \frac{\partial u(x_1, x_2)}{\partial x_2}\dot{x}_2 \tag{7}$$

or:

$$\frac{du(x_1, x_2)}{dt} = \frac{\partial u(x_1, x_2)}{\partial x_1}f_1(x_1, x_2) + \frac{\partial u(x_1, x_2)}{\partial x_2}f_2(x_1, x_2) = \overline{\vec{V}_u \times \vec{\dot{X}}} \tag{8}$$

where:

$$\overline{\vec{V}_u \times \vec{\dot{X}}} = \left|\vec{V}_u\right|.\left|\vec{\dot{X}}\right|\sin(\alpha)$$

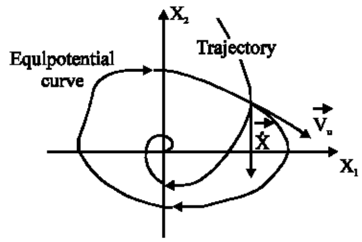

Fig. 1: Trajectories of an asymptotic stable system

and α is the angle between $\vec{V}_u$ and $\dot{\vec{X}}$. So the inequality 2 can be written as:

$$\overline{\vec{V}_u \times \dot{\vec{X}}} < 0 \tag{9}$$

and this means that the direction of the trajectories of the system 1 are to inside of the equipotential curves $u(x_1, x_2) = C$ as shown in Fig. 1, on the other hand $C \in R$ and C can be changed so that closed curves $u(x_1, x_2) = C$ enclosing the equilibrium point could tend to be smaller and smaller and finally approach to the equilibrium point (the origin). This means that the direction of the trajectory of the system 1 will tend to the origin, so the system 1 is asymptotically stable.

Theorem 2: The second-order autonomous system described by Eq. 1 is unstable around the equilibrium point (X = 0) if there are equipotential curves $u(x_1, x_2) = C$ with clockwise direction, enclosing the equilibrium point and further on the trajectories of the system 1:

$$\frac{du(x_1, x_2)}{dt} > 0 \tag{10}$$

Proof: It follows from the proof of the theorem 1 that inequality 10 means that:

$$\overline{\vec{V}_u \times \dot{\vec{X}}} > 0 \tag{11}$$

and in the similar manner with the proof of the theorem 1, the direction of the trajectories of the system 1 are to outside of the equipotential curves $u(x_1, x_2) = C$ as shown in Fig. 2, so the trajectories tend to infinity (far and farther of the equilibrium point) and this means that the system around the equilibrium point (X = 0) is unstable.

Definition 4: A limit cycle is said asymptotic stable if all trajectories in vicinity of the limit cycle converge to it as $t \to \infty$. Otherwise the limit is semi-stable or unstable (Doyle *et al.*, 1992).

Theorem 3: Consider the second-order autonomous system 1, suppose that no equilibrium point belongs to the compact set M which encloses the origin (X = 0). There are equipotential curves $u(x_1, x_2) = C_1$ and $u(x_1, x_2) = C_2$ with clockwise directions that belong to M, does not intersect one with another, enclose the origin and satisfy the following inequalities on the trajectories of the system 1:

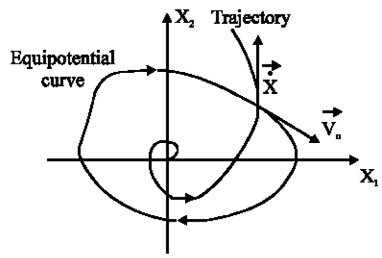

Fig. 2: Trajectories of an unstable system

$$\frac{du_1(x_1, x_2)}{dt} \geq 0 \tag{12}$$

$$\frac{du_2(x_1, x_2)}{dt} \leq 0 \tag{13}$$

if and only if there exists an asymptotic stable limit cycle L so that:

$$L \subset \text{int}\Omega \tag{14}$$

where, $\Omega(C_1, C_2)$ is the region located between $u_1(x_1, x_2) = C_1$ and $u_2(x_1, x_2) = C_2$.

Proof: (⇒) Similar to the theorem 1, the inequality 12 can be rewritten as:

$$\overline{\vec{V}_{u_1} \times \dot{\vec{X}}} \geq 0 \tag{15}$$

This means that the direction of the trajectories of the system 1 is to the outside of the equipotential curves $u_1(x_1, x_2) = C_1$ as shown in Fig. 3. In the similar manner the inequality (13) can be expressed as:

$$\overline{\vec{V}_{u_2} \times \dot{\vec{X}}} \leq 0 \tag{16}$$

where, $\vec{v}_{u_2}$ is the velocity vector on the $u_2(x_1, x_2) = C_2$ and this means that the direction of the trajectories of the system 1 is to the inside of the equipotential curves $u_1(x_1, x_2) = C_2$ as shown in Fig. 3. On the other hand there is no equilibrium points belonging to M and consequently to $\Omega(C_1, C_2)$, so there is an asymptotic stable limit cycle L so that $L \subset \text{int}\Omega$.

(⇒) the necessary condition can similarly be proofed using above geometric concepts.

Remark 1: If on the trajectories of the system 1:

$$\frac{du_1(x_1, x_2)}{dt} = 0$$

or:

$$\frac{du_2(x_1, x_2)}{dt} = 0$$

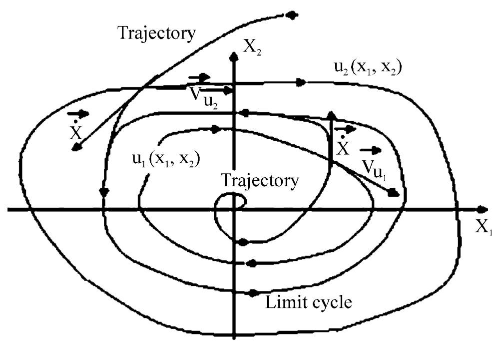

Fig. 3: A stable limit cycle appearing in the system

the equipotential curve $u_1(x_1, x_2) = C_1$ or $u_2(x_1, x_2) = C_2$ itself is the limit cycle respectively.

Example 1: Consider the following system:

$$\begin{cases} \dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2 \\ \dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2) \end{cases}$$

By choosing:

$$u(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) = C$$

for C<1, it can be seen that not only the equipotential curves $u(x_1, x_2) = C$ are closed but also on the trajectories of the system:

$$\frac{du(x_1, x_2)}{dt} < 0$$

so the system is asymptotic stable.

Example 2: Consider the following system:

$$\begin{cases} \dot{x}_1 & = & e^{x_1}\sin(x_2) - 4x_1^3 \\ \dot{x}_2 & = & e^{x_2}\sin(x_1) + x_2^{\ 7} \end{cases}$$

By choosing:

$$u(x_1,\ x_2) = -\frac{1}{2}x_1^{-2} - \frac{2}{3}x_2^{-6} = C$$

for all value of the C, it can be seen that not only the equipotential curves $u_1(x_1, x_2) = C$ are closed but also on the trajectories of the system:

$$\frac{du(x_1,\ x_2)}{dt} > 0$$

so the system is unstable.

Example 3: Consider the following system

$$\begin{cases} \dot{x}_1 = x_2 - x_1^7\,(x_1^4 + 2x_2^2 - 10) \\ \dot{x}_2 = -x_1^3 - 3x_2^2\,(x_1^4 + 2x_2^2 - 10) \end{cases}$$

By choosing:

$$u_1\,(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 = C_1$$

for $0<C_1<2.5$, it can be seen that not only the equipotential curves $u_1\ (x_1,\ x_2) = C_1$ are closed but also we have:

$$\frac{du_1\,(x_1,\ x_2)}{dt} > 0$$

Also by choosing:

$$u_2\,(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 = C_2$$

for $2.5<C_2$, it can be seen that not only the equipotential curves $u_2\ (x_1,\ x_2) = C_2$ are closed but also:

$$\frac{du_2\,(x_1,\ x_2)}{dt} < 0$$

so there is an asymptotic stable limit cycle located between the $u_1\ (x_1,\ x_2) = C_1$ and $u_2\ (x_1,\ x_2) = C_2$. The area located between the $u_1\ (x_2,\ x_2) = C_1$ and $u_2\ (x_1,\ x_2) = C_2$ is an invariant set as the following set:

$$\Omega\,(C_1, C_2) = \left\{(x_1,\ x_2) \middle| C_1 < \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 < C_2\right\} \tag{17}$$

where, $0<C_1<2.5<C_2$. It is clear that the limit cycle can be estimated by varying the C_1 and C_2 in (17). In above set by increasing C_1 and decreasing C_2, the limit cycle can be found as:

$$\frac{1}{4}x_2^2 + \frac{1}{2}x_2^2 = 2.5$$

FIRST APPLICATION-CONTROL OF LIMIT CYCLES

Consider the following nonlinear autonomous system:

$$\begin{cases} \dot{x}_1 = f_1\,(x_1,\ x_2,\ u_1^*) \\ \dot{x}_2 = f_2\,(x_1,\ x_2,\ u_2^*) \end{cases} \tag{18}$$

where, u_1^* and u_2^* are the control inputs as the form of state feedback presented by the following equations.

$$\begin{cases} u_1^* = h_1(x_1,\ x_2) \\ u_2^* = h_2(x_1,\ x_2) \end{cases} \tag{19}$$

Now, the question is that how $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ must be chosen so that an asymptotic stable limit cycle can be added to the system 18 The condition:

$$\frac{du_1(x_1, x_2)}{dt} \geq 0$$

on the trajectories of the system 18 in the theorem 3 can be rewritten as following inequality:

$$\frac{du_1(x_1, x_2)}{dx_1} f_1(x_1, x_2, u_1^*) + \frac{du_1(x_1, x_2)}{dx_2} f_2(x_1, x_2, u_2^*) \geq 0 \tag{20}$$

and in the similar manner the:

$$\frac{du_2(x_1, x_2)}{dt} \leq 0$$

appeared in the theorem 3, can be expressed as:

$$\frac{du_2(x_1, x_2)}{dx_1} f_1(x_1, x_2, u_1^*) + \frac{du_2(x_1, x_2)}{dx_2} f_2(x_1, x_2, u_2^*) \leq 0 \tag{21}$$

The inequalities 20 and 21 give the conditions which have to be satisfied by u^*_1, u^*_2, $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in order to appear an asymptotic limit cycle in the system 18.

Example 4: Consider the following system:

$$\begin{cases} \dot{x}_1 = x_2^7 - x_1^3 + u_1^* \\ \dot{x}_2 = -x_1 - x_1^2 x_2 + u_2^* \end{cases} \tag{22}$$

It is clear that the equilibrium point at the origin is asymptotic stable. Now, the state feedback lows (u_1* and u_2*) have to be determined so that an asymptotic stable limit cycle can be added to the resulted closed loop system. By choosing equipotential curves as:

$$u_1(x_1, x_2) = 4x_1^2 + x_2^8 = C_1;\ 0 < C_1 < 12 \tag{23}$$

$$u_2(x_1, x_2) = 4x_1^2 + x_2^8 = C_2; 14 < C_2 \tag{24}$$

and replacing Eq. 23-24 in inequalities 20-21, respectively the following inequalities are found:

$$8x_1(x_2^7 - x_1^3 + u_1^*) + 8x_2^7(-x_1 - x_1^2 x_2 + u_2^*) \geq 0;\ \text{for } 0 < 4x_1^2 + x_2^8 < 12 \tag{25}$$

and

$$8x_1(x_2^7 - x_1^3 + u_1^*) + 8x_2^7(-x_1 - x_1^2 x_2 + u_2^*) \leq 0;\ \text{for } 14 < 4x_1^2 + x_2^8 \tag{26}$$

It can be derived from inequalities 25-26 that:

$$-8x_1^4 - 8x_1^2 x_2^8 + 8x_1 u_1^* + 8x_2^7 u_2^* \geq 0;\ \text{for } 0 < 4x_1^2 + x_2^8 < 12 \tag{27}$$

and

$$-8x_1^4 - 8x_1^2 x_2^8 + 8x_1 u_1^* + 8x_2^7 u_2^* \leq 0;\ \text{for } 14 < 4x_1^2 + x_2^8 \tag{28}$$

By choosing the state feedback laws as the following forms:

$$\begin{cases} u_1^* = h_1(x_1, x_2) = x_1(\beta + \frac{3}{4}x_2^8) \\ u_2^* = h_2(x_1, x_2) = 0 \end{cases} \tag{29}$$

and by replacing Eq. 29 in inequalities 27-28, the following inequalities are found as the conditions to appear an asymptotic limit cycle in the system:

$$-2x_1^2(4x_1^2 + x_2^8 - 4\beta) \geq 0;\ \text{for } 0 < 4x_1^2 + x_2^8 < 12 \tag{30}$$

and

$$-2x_1^2(4x_1^2 + x_2^8 - 4\beta) \leq 0;\ \text{for } 14 < 4x_1^2 + x_2^8 \tag{31}$$

The inequalities 30-31 both are satisfied, when:

$$3 \leq \beta \leq \frac{14}{3} \tag{32}$$

It also follows from the theorem 3 that the asymptotic stable limit cycle L, which is added to the system 18 using state feedback, appears in the following region:

$$L \subset \left\{(x_1, x_2) \middle| 12 \leq 4x_1^2 + x_2^8 \leq 14\right\} \tag{33}$$

SECOND APPLICATION-DESIGN OF OSCILLATION IN ELECTRONIC CIRCUITS

Consider the basic model of an oscillator shown in Fig. 4 including LC tank and dependent current source. The state equations of the circuit can be written as the following autonomous system:

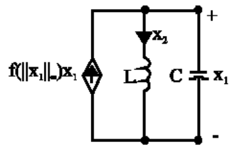

Fig. 4: Basic circuit of an oscillator including LC tank

$$\begin{cases} \dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}u_1^* \\ \dot{x}_2 = \frac{1}{L}x_1 \end{cases} \tag{34}$$

where, x_1 and x_2 are defined as the voltage which appears across the capacitor and the current of the inductor respectively. As we see the dependent current source plays the role of the control input. By considering the Eq. 19 and defining the state feedback as the following equation:

$$u^*_1 = f(\|x_1\|\infty)x_1 \tag{35}$$

the Eq. 34 can be rewritten as:

$$\begin{cases} \dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}f(\|x_1\|_\infty)x_1 \\ \dot{x}_2 = \frac{1}{L}x_1 \end{cases} \tag{36}$$

where, $\|\cdot\|\infty$ is infinite norm and $f(\cdot)$ is determined by the electronic elements such as BJT, MOSFET and used to design the oscillator. Now, the equipotential curves $u_1(x_1, x_2) = C_1$ and $u_2(x_1, x_2) = C_2$ are considered as the circles surrounding the origin and described by the following equations:

$$u_1(x_1, x_2) = \frac{1}{L}x_1^2 + \frac{1}{C}x_2^2 = C_1 \tag{37}$$

$$u_2(x_1, x_2) = \frac{1}{L}x_1^2 + \frac{1}{C}x_2^2 = C_2 \tag{38}$$

where, $C_1 \leq C_2$. By checking inequalities 12-13 of the theorem 3, we have:

$$\frac{du_1(x_1, x_2)}{dt} = \frac{2}{LC}f(\|x_1\|_\infty)x_1^2 \geq 0 \tag{39}$$

$$\frac{du_2(x_1, x_2)}{dt} = \frac{2}{LC}f(\|x_1\|_\infty)x_1^2 \leq 0 \tag{40}$$

It is clear that above inequalities can not both be satisfied unless:

$$f(\|x_1\|\infty) = 0 \tag{41}$$

This is the necessary and sufficient condition to appear oscillation in the circuit.

Suppose that the oscillation appearing in the circuit is as the form of sinusoidal wave, in other word the voltage appearing across the capacitor of the tank is expressed by the following equation:

$$x_1 = V_m \cos(\omega_s t) \tag{42}$$

It is clear that:

$$\omega_s = \frac{1}{\sqrt{LC}} \tag{43}$$

and

$$\|x_1\|_\infty = \sup|V_m \cos(\omega_s t)| = V_m \tag{44}$$

So the Eq. 41 can be rewritten as:

$$f(\|x_1\|_\infty) = f(V_m) = 0 \tag{45}$$

CONCLUSION

In this study, the behavior of a second-order nonlinear dynamical system around its equilibrium point was analyzed based on the behavior of some appropriate equipotential curves which were considered around the same equilibrium point. In fact two sets of equipotential curves were considered so that a set of the equipotential curves had a role as the upper band of the system trajectory and another set played a role as the lower band. It was shown that stability of the system around its equilibrium point can be assessed using the behavior of these two set of equipotential curves. It was shown that the proposed mathematical method can detect a stable limit cycle appearing in the systems and also the method has some useful applications that described in section 4 and 5. For future works the method can be extended for analyzing of third order dynamical systems.

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