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Capability of Satisfying Boundary Conditions in Various Velocity and Temperature Profiles and its Effect on the Key Boundary Layer Parameters in Integral Method

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ABSTRACT
In this study, a further investigation on the integral solution of the boundary layer momentum and energy equations which are originally concerned by von Karman and Pohlhausen, is done. According to various profiles that we suggest, an analysis on the errors in the values of the friction coefficient, C_f and Nusselt number, Nu, are drawn. Although, it is implied in references that the best choose for the velocity and temperature profiles is the one that can satisfy necessary and sufficient boundary conditions, but several combinations for degree of velocity and temperature profiles exist that show better approximations for values of Nu compare with mentioned profiles.

Key words: Boundary layer, Blasius solution, momentum and energy integral equations, Pohlhausen method, friction coefficient, nusselt number

INTRODUCTION
Despite its simplicity, parallel flow over a flat plate occurs in numerous engineering applications. Assuming steady, incompressible, laminar flow with constant fluid properties and negligible body forces, viscous dissipations and without any heat generation and pressure gradient, the boundary layer equations reduce to:

Continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

Momentum:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

Energy:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

For solving these partial differential equations, with an ingenious coordinate transformation, Blasius and Pohlhausen showed that the velocity profile u/u_∞ and temperature profile:

$$\frac{T-T_s}{T_\infty-T_s}$$

remain geometrically similar. These similarities are of the functional forms:

$$\frac{u}{u_\infty}=f\left(\frac{y}{\delta}\right)$$

and

$$T^*=\frac{T-T_s}{T_\infty-T_s}=g\left(\frac{y}{\delta_t}\right)$$

(Bergman *et al.*, 2011).

The final results of the Blasius solution lead to the following (Bergman *et al.*, 2011):

$$\delta=\frac{5x}{Re_x^{1/2}},\ C_{f,x}=\frac{0.664}{Re_x^{1/2}},\ \zeta=Pr^{-1/3}\ \text{and}\ Nu_x=0.332Re_x^{1/2}\,Pr^{1/3} \tag{4}$$

An alternative approach to solving the boundary layer equations involves the use of an approximate integral method. The approach was originally proposed by von Karman and applied by Pohlhausen. It is without the mathematical complications inherent in the exact method; yet it can be used to obtain reasonably accurate results for the key boundary layer parameters (δ, C_f, δ_t, Nu). To use the method, the boundary layer equations, must be cast in integral forms. These forms are obtained by integrating the Eq. 1-3 in the y-direction across the boundary layer that leads to following equations:

$$\frac{d}{dx}\left[\int_0^{\delta}(u_\infty-u)\,u\,dy\right]=\nu\left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{5}$$

$$\frac{d}{dx}\left[\int_0^{\delta_t}(T_\infty-T)\,u\,dy\right]=\alpha\left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{6}$$

which are the integral forms of the boundary layer momentum and energy equations.

These integral equations can be used to obtain approximate boundary layer solutions. The procedure involves first assuming reasonable functional forms for the unknowns u and T in terms of the corresponding (unknown) boundary layer thicknesses. The assumed forms must satisfy appropriate boundary conditions. Substituting these forms into the integral equations, expressions for the boundary layer thicknesses may be determined and the assumed functional forms may then be completely specified. Although, this method is approximate, it frequently leads to accurate results for the surface parameters.

Although, there are a wide range of studies that apply the integral boundary layer equation method but there are not enough explanation on the effect of the various types of velocity and temperature profiles on the boundary layer parameters.

A new investigation on the integral solution of momentum equation is carried out by (Saedodin and Barforoush, 2012). By various profiles that they suggested, an analysis on the errors for the values of the boundary layer thickness and friction coefficient, was carried out. The problematic conclusion is that, may be a chosen velocity profile that is nearly lies to the exact one, creates bigger errors in the values of δ and C_f comparing with the velocity profile that has more derivate from the exact profile.

This study emphasizes on the effect of the velocity and temperature profile types on the boundary layer parameters such as friction factor and Nusselt number.

Obviously, if we would choose profiles in the forms of polynomials in n degree, we need to n+1 conditions to evaluate the coefficients of those functions.

For the flow over a standstill plate with a constant temperature, two following statements are apparent:

I u/u_∞ and T^* must be zero at $y = 0$ that come from no-slip condition and the continuity of temperature in the surface, respectively

II u/u_∞ at $y = \delta$ and T^* at $y = \delta_t$ must be unity that come from the concept of the boundary layer thicknesses

Then the following boundary conditions must be applied step by step for each one order that the degree of polynomials is increased:

III $\partial u/\partial y$ and $\partial T^*/\partial y$ must be zero at $y = \delta$ and $y = \delta_t$, respectively

IV $\partial^2 u/\partial y^2$ and $\partial^2 T^*/\partial y^2$ must be zero at $y = 0$, that come from Eq. 2 and 3 and considering no-slip condition in the surface

In fact, four above statements are the whole of physical conditions that the suggested polynomials must satisfy. But, for polynomials that their degrees are higher than 3, we need to extra boundary conditions. In this article we choose:

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=\delta} = \left.\frac{\partial^2 T^*}{\partial y^2}\right|_{y=\delta_t} = 0$$

for polynomials of degree higher than 3 and:

$$\left.\frac{\partial^3 u}{\partial y^3}\right|_{y=\delta} = \left.\frac{\partial^3 T^*}{\partial y^3}\right|_{y=\delta_t} = 0$$

for polynomials of degree 4. The recent assumptions and also the statement number III, denote to uniformity of velocity and temperature distributions out of the hydrodynamic and thermal boundary layers, respectively.

Applying these boundary conditions lead to the following functionals that are listed by increasing the degree of polynomials:

$$\frac{y}{\varepsilon} \tag{7}$$

$$2\frac{y}{\varepsilon}-\left(\frac{y}{\varepsilon}\right)^2 \tag{8}$$

$$\frac{3}{2}\frac{y}{\varepsilon}-\frac{1}{2}\left(\frac{y}{\varepsilon}\right)^3 \tag{9}$$

$$2\frac{y}{\varepsilon}-2\left(\frac{y}{\varepsilon}\right)^3+\left(\frac{y}{\varepsilon}\right)^4 \tag{10}$$

$$\frac{5}{2}\frac{y}{\varepsilon}-5\left(\frac{y}{\varepsilon}\right)^3+5\left(\frac{y}{\varepsilon}\right)^4-\frac{3}{2}\left(\frac{y}{\varepsilon}\right)^5 \tag{11}$$

where, ε must be chosen as δ and δ_t for velocity and temperature profiles, respectively.

DESCRIPTION OF THE PROBLEM

As we suggest five optional profiles for velocity and temperature, then all of their combinations create 25 cases. We nominate these cases by symbol M-N that M and N refer to degrees of velocity and temperature polynomials, respectively.

By using each of these combinations into Eq. 5 and 6 and by considering that the value of δ and δ_t are zero in the leading edge x = 0 we obtain the values of δ, C_f, ζ, Nu and the errors of calculated values of C_f and Nu which are calculated in the following manner:

$$\text{Error in percent}=\frac{\text{(actual value-estimated value)}}{\text{actual value}}\times 100$$

The results of these calculations are tabulated in Table 1 and are drawn in Fig. 1-6.

A sample calculation for case 2-4 is presented in the following:

Substituting:

$$\frac{u}{u_\infty}=2\frac{y}{\delta}-\left(\frac{y}{\delta}\right)^2$$

in Eq. 5, we have:

$$\frac{d}{dx}\left[\int_0^\delta u_\infty^2\left(1-2\frac{y}{\delta}+\left(\frac{y}{\delta}\right)^2\right)\left(2\frac{y}{\delta}-\left(\frac{y}{\delta}\right)^2\right)dy\right]=\frac{2\nu}{\delta}u_\infty$$

Table 1: Results of the errors for values of C_f and Nu with different velocity and temperature profiles

Profile type M-N	$\delta\times Re^{1/2}/x$	$C_f\times Re^{1/2}$	$\zeta\times Pr^{1/3}$	$Nu\times Re^{-1/2}\times Pr^{-1/3}$	Error C_f (%)	Error Nu (%)
1-1	3.464	0.577	1.000	0.289	-13.1	-13
1-2			1.587	0.364		+9.6
1-3			1.357	0.319		-3.9
1-4			1.710	0.338		+1.8
1-5			2.061	0.350		+5.4
2-1	5.477	0.730	0.585	0.312	+9.9	-6
2-2			0.928	0.393		+18.3
2-3			0.794	0.345		+3.9
2-4			1.000	0.365		+9.9
2-5			1.205	0.379		+14.2
3-1	4.641	0.646	0.719	0.300	-2.7	-9.6
3-2			1.141	0.378		+13.9
3-3			0.976	0.331		-0.3
3-4			1.229	0.351		+5.7
3-5			1.481	0.364		+9.6
4-1	5.836	0.685	0.561	0.306	+3.2	-7.8
4-2			0.890	0.385		+16
4-3			0.761	0.338		+1.8
4-4			0.959	0.358		+7.8
4-5			1.155	0.371		+11.7
5-1	7.036	0.711	0.624	0.570	+7.1	+71.7
5-2			0.786	0.452		+36.1
5-3			0.739	0.481		+44.9
5-4			0.846	0.420		+26.5
5-5			0.947	0.375		+13

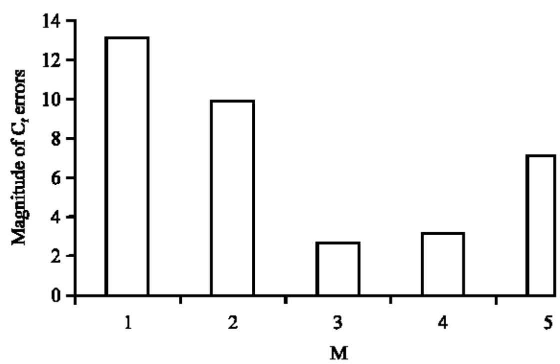

Fig. 1: Effect of velocity polynomial degrees on the magnitude of C_f errors

Applying the integration leads to:

$$\frac{d}{dx}\left[u_\infty \frac{2}{15}\delta\right] = \frac{2\nu}{\delta}$$

By separating of the variables:

$$\delta\, d\delta = \frac{15\nu}{u_\infty} dx$$

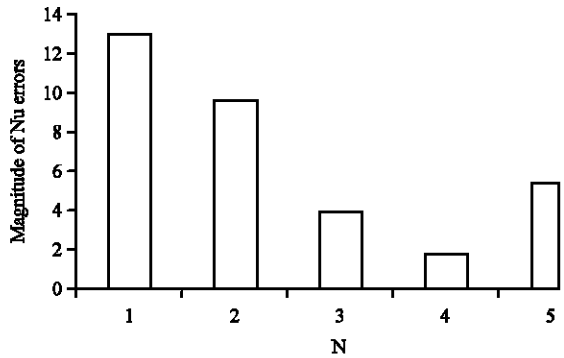

Fig. 2: Effect of temperature polynomial degrees on the magnitude of Nu errors (the degree of velocity polynomial is equal to 1)

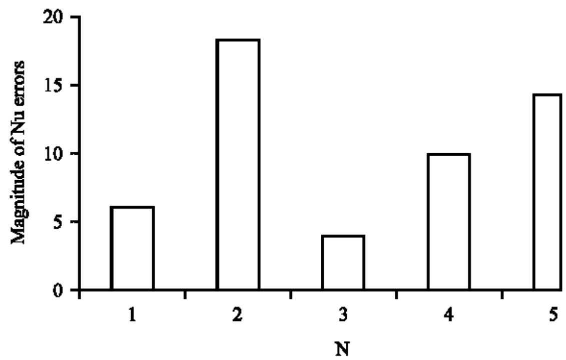

Fig. 3: Effect of temperature polynomial degrees on the magnitude of Nu errors (the degree of velocity polynomial is equal to 2)

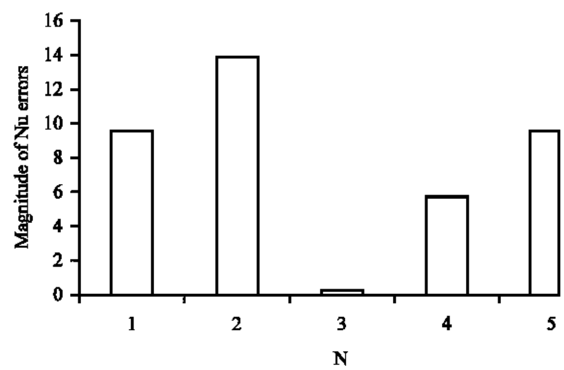

Fig. 4: Effect of temperature polynomial degrees on the magnitude of Nu errors (the degree of velocity polynomial is equal to 3)

By solving this differential equation and applying the boundary condition $\delta(0) = 0$, $\delta(x)$ is calculated as:

$$\delta = \frac{5.477x}{Re_x^{1/2}}$$

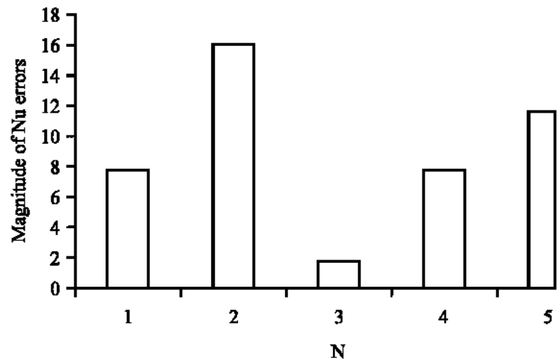

Fig. 5: Effect of temperature polynomial degrees on the magnitude of Nu errors (the degree of velocity polynomial is equal to 4)

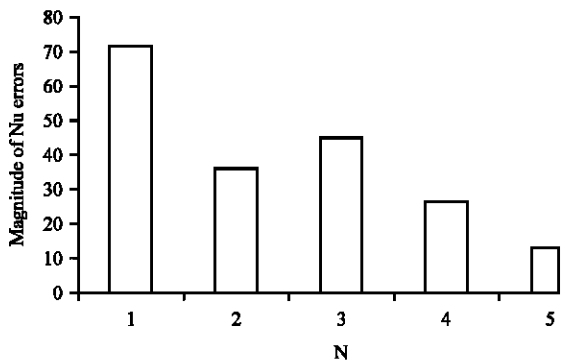

Fig. 6: Effect of temperature polynomial degrees on the magnitude of Nu errors (the degree of velocity polynomial is equal to 5)

Using this value together with:

$$C_{f,x} = \frac{\mu \left.\frac{\partial u}{\partial y}\right|_{y=0}}{\frac{1}{2}\rho u_\infty^2}$$

$C_{f,x}$ is calculated as:

$$C_{f,x} = \frac{0.730}{Re_x^{1/2}}$$

Now, using the above mentioned for u/u∞ and:

$$T^* = 2\frac{y}{\delta_t} - 2\left(\frac{y}{\delta_t}\right)^3 + \left(\frac{y}{\delta_t}\right)^4$$

in Eq. 6 we have:

$$\frac{d}{dx}\left[\int_0^{\delta_t} u_\infty \left(T_s - T_\infty\right)\left(1 - 2\frac{y}{\delta_t} + 2\left(\frac{y}{\delta_t}\right)^3 - \left(\frac{y}{\delta_t}\right)^4\right)\left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right) dy\right] = \frac{2\alpha}{\delta_t}\left(T_s - T_\infty\right)$$

Applying the integration leads to:

$$\frac{d}{dx}\left[u_\infty \left(T_s - T_\infty\right)\left(\frac{1}{210}\frac{\delta_t^2\left(-5\delta_t + 28\delta\right)}{\delta^2}\right)\right] = \frac{2\alpha}{\delta_t}\left(T_s - T_\infty\right)$$

By assuming that the thermal boundary layer is thinner than the hydrodynamic boundary layer ($\delta_t<\delta$) then by introducing a variable such as $\zeta = \delta_t/\delta$ we have $\zeta<1$ and the term involving ζ^4 is small compared with the ζ^2 term. By neglecting the ζ^4 term, the last equation will be reduced to the following:

$$\frac{d}{dx}\left[u_\infty \frac{2}{15}\delta\zeta^2\right] = \frac{2\alpha}{\delta\zeta}$$

Using:

$$\delta = \frac{5.477x}{Re_x^{1/2}}$$

and by rearranging, we obtain:

$$\zeta^3 + 4x\zeta^2\frac{d\zeta}{dx} = \frac{\alpha}{\nu}$$

by introducing $\zeta^3 = \varepsilon$, the above nonlinear differential equation could be transformed to the following linear first order differential equation:

$$\frac{d\varepsilon}{dx} + \frac{3}{4x}\varepsilon = \frac{3\alpha}{4\nu x}$$

By solving this differential equation and applying the boundary condition $\zeta(0) = 0$, $\zeta(x)$ is calculated. And finally, using this value together with:

$$\delta = \frac{5.477x}{Re_x^{1/2}}$$

and

$$Nu_x = -\frac{x}{k}\frac{k\left.\frac{\partial T}{\partial y}\right|_{y=0}}{T_s - T_\infty}$$

Nu_x is calculated as $Nu_x = 0.365\, Re_x^{1/2}\, Pr^{1/3}$.

CONCLUSION

The following results can be obtained from Table 1:

- As M rises from 1 to 3, the errors for the values of C_f decrease and as M rises from 3 to 5 these errors increase. On the other hand, the best estimation for the value of C_f occur in M = 3. For justification of this result in the references (White, 1998) it is said that "as the third degree polynomial profile for velocity satisfies four boundary conditions that are physical nature of the problem, it leads to smallest errors for C_f in compare with another ones"
- The errors for the determined values of C_f in various cases are listed by ascending arrangement
  3-N 4-N 5-N 2-N 1-N
  That shows the importance of satisfying appropriate boundary conditions in velocity profile for evaluating C_f. Simply, even if redundant boundary conditions are applied it is better to vanish some of the boundary conditions
- For equal degree of velocity and temperature profiles the errors for the determined values of Nu are listed by ascending arrangement
  3-3 4-4 5-5 2-2 1-1
  And this shows although in the cases 5-N the choices N = 1 to 4 lead to big errors for Nu, but in these forms, choice N = 5 gives acceptable error for this parameter
- Based on the mathematics principals, every inaccuracy on the used profile
  - Could be stylized when it is used in integration procedures and then will create negligible errors
  - Could be amplified when it is used in derivation procedures and then will create considerable errors

  Since in the procedure for calculation of Nu the velocity profile is used in both of momentum and energy equation unlike the temperature profile which is used only in energy equation, often when x>y the case y-x provide the better estimations for Nu compare with the case x-y
- Except for case 1-1, the other profiles 1-N often lead to a very small errors for the value of Nu and this clarify that the crude linear profile may be used to estimate the Nu very better compare to several types of the other velocity distribution. In the other words, failure to satisfy the boundary conditions (III) and (IV) in the velocity profile often does not lead to considerable error on the value of Nu
- Although, the calculated values of C_f in cases N = 1 and 3 are lower and in cases N = 2, 4 and 5 are upper than exact value of C_f, but there is no similar conclusion for Nu, except for case M = 5. Put another way, in each degree for velocity profile, various temperature profiles lead to lower or upper values of Nu comparing the exact values, except for M = 5 that in this case all of the values for Nu are bigger from the exact value
- Generally, it seems that the cases 5-N have the worst estimation for the value of Nu and it implies that when the velocity profile is chosen in fifth degree, that has two additional boundary conditions further than basic boundary conditions I to IV, it will lead to the drastic errors on the values of Nu
- It is obvious that the best estimations for the values of C_f and Nu can be obtained simultaneously in the case 3-3 that is conformed the Pohlhausen solution. In fact, it implies that since the third degree polynomials for the velocity and temperature can satisfy all of the necessary and sufficient boundary condition I to IV then this option will lead to the best approximation (Kays and Crawford, 1993). But it is important to notice that there are many cases in form 3-N that are not better than M-N (i.e., see cases 3-1 and 2-1) and so there are

many cases in form M-3 that are not better than M-N (i.e., see cases 1-3 and 1-4). In the other words, ideality of third degree polynomial is only when we choose M and N equal to 3 simultaneously. And there are many cases that the choice of just one of M or N equal to 3, which does not provide satisfactory estimation for Nu

NOMENCLATURE

u	x velocity component, ms^{-1}
v	y velocity component, ms^{-1}
u_∞	Free stream velocity, ms^{-1}
T	Temperature, K
T_∞	Free stream temperature, K
T_s	Surface temperature, K
Pr	Prandtl number
Re_x	Local Reynolds number
$C_{f,x}$	Local friction coefficient
Nu_x	Local Nusselt number

Greek symbols:

α	Thermal diffusivity, m^2s^{-1}
ν	Kinematic viscosity, m^2s^{-1}
δ	Hydrodynamic boundary layer thickness, m
δ_t	Thermal boundary layer thickness, m
ζ	The ratio of δ_t/δ

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