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Research Article
Some Structures to Linear and Nonlinear Schrödinger Equations Via the Differential Transform Method

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Abstract
Background and Objective: The nonlinear Schrödinger equation plays an important role in Physics and Applied Mathematics as well. The analysis of structures of the Schrödinger equation has gained considerable momentum and a particular attention. This study aimed to use the symbolic solutions via the differential transform approach. **Methodology:** Through differential transform method, some exact and approximate traveling wave solutions of linear and nonlinear Schrödinger equations are investigated. **Results:** Miscellaneous traveling wave solutions including, exponential solutions are obtained. On one side, for the first case study, two exact expressions of solutions for the linear case are obtained with the iterated sequence. On the other side, for the second case study, four exact expressions of solutions for the nonlinear case have been computed exactly as in the linear case. **Conclusion:** Some examples are examined, with different physical structures to show the real power of the proposed method. Through computational aspects, this approach can successfully ensure the convergence of true solutions and gives new results under a symbolic aspect.

Key words: Differential transform, nonlinear equation, Schrödinger equation, nonlinear wave, symbolic computation, travelling wave

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Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

Considerable efforts have been undertaken in recent years towards obtaining the analytical solutions of wave equations for certain potentials of physical interest[1-6]. There are relatively few quantum mechanical problems for which the linear and nonlinear Schrödinger equations are exactly solvable[7-18]. So far, many explicit methods have been developed to solve linear and nonlinear wave equations. Some of them are Darboux transformation, Cole-Hopf transformation, Painleve method, homogeneous balance method, tanh method, sine-cosine method and so on. A great deal of approximate schemas and numerical approaches have also appeared to calculate the quantum quantities of the Schrödinger equation for numerous situations[19-26]. The scope of this area remains until now a more active field of diverse problems.

The main part of this work serves to the construction of specific solutions by a differential transformation technique for the Schrödinger equation problem. Specifically, the Taylor expansion is the principal argument on the construction of the differential transformation method (hereafter called DTM). Exact solutions can be achieved by the known forms of the truncated series solutions. The closed form series solutions or in the form of a polynomial can be obtained and are fast compared with the solutions that are calculated from other approaches.

BASIC DEFINITIONS OF THE DTM

Some definitions and properties of the two-dimensional differential transform method that may be found by Zhou[27], Ayaz[28-29] and Kurnaz *et al.*[30]. The main issues of the method summarized as follows: W (k, h) call as the differential transformed function and w(x, t) as the differential inverse transform.

Definition 1: The two-dimensional differential transform of function W (k, h) is given by:

$$W(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}w(x,t)}{\partial x^k\partial t^h}\right]_{x=x_0,\,t=t_0} \quad (1)$$

where, w(x, t) is analytic and differentiated continuously with respect to x and t in the domain of interest. It is noted that, w(x, t) represents the original function.

Definition 2: The differential inverse transform of W (k, h) is given in Taylor's series form:

Table 1: Some transformed functions with the corresponding original forms

Original function w(x, t)	Transformed function W (k, h)
$u(x,t)\pm v(x,t)$	$U(k,h)\pm V(k,h)$
$\alpha u(x,t)$	$\alpha U(k,h)$
$\frac{\partial^{r+s}u(x,t)}{\partial x^r\partial t^s}$	(k+1) (k+2)...(k+r) (h+1) (h+2)...(h+s) U(k+r, h+s)
u(x, t) v(x, t)	$\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$
u(x, t) v(x, t) g(x, t)	$\sum_{n=0}^{k}\sum_{m=0}^{k-n}\sum_{j=0}^{h}\sum_{l=0}^{h-j}U(n,h-j-l)V(m,j)G(k-n-m,l)$

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}w(x,t)}{\partial x^k\partial t^h}\right]_{x=x_0,\,t=t_0}(x-x_0)^k(t-t_0)^h \quad (2)$$

In most cases (x_0, t_0) are taken (0, 0) and the practical expression of w(x, t) is shown as:

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^k t^h \quad (3)$$

Definition 3: It was assumed that, the convergence of the Taylor's series must be satisfied for both particular values of n and m as:

$$w(x,t)=\sum_{k=0}^{n}\sum_{h=0}^{m}W(k,h)x^k t^h \quad (4)$$

Now, let us list some properties in Table 1, which will be useful in the next development.

STRUCTURES FOR SOME QUANTUM MODELS

In this section, the method with two examples is explored. First, an application in one-dimension with a linear problem that can be useful for many quantum situations is introduced. It is always required to add the boundary conditions. It is also necessary to specify some information about the initial conditions to determine the solutions at a later time.

Example 1: Consider the following standard linear Schrödinger equation:

$$w_t+\eta i w_{xx}=0 \quad (5)$$

where, the subscripts in w_t and w_{xx} denote the partial derivatives and w(x, t) is a sufficiently differentiable function, with the initial condition $w(x,0)=e^{iax}$. The Eq. 5 provides two types of solutions according to the values of $\eta=\pm1$.

Taking the differential transform method to the Eq. 5 together with its initial condition, yields the following relation:

$$(h+1)\,W(k,h+1)+\eta i(k+1)(k+2)\,W(k+2,h)=0 \quad (6)$$

rearrange Eq. 6 the recurrence relation is obtained:

Table 2: Presents some values of W(k, h) for $\eta = \pm 1$

k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)
0 1	ia^2	1 1	$-a^3$	2 1	$-\frac{ia^4}{2}$	3 1	$\frac{a^5}{6}$	4 1	$\frac{ia^6}{24}$	5 1	$-\frac{a^7}{120}$
0 2	$-\frac{a^4}{2}$	1 2	$-\frac{ia^5}{2}$	2 2	$\frac{a^6}{4}$	3 2	$\frac{ia^7}{12}$	4 2	$-\frac{a^8}{48}$	5 2	$-\frac{ia^9}{240}$
0 3	$-\frac{ia^6}{6}$	1 3	$\frac{a^7}{6}$	2 3	$\frac{ia^8}{12}$	3 3	$-\frac{a^9}{36}$	4 3	$-\frac{ia^{10}}{144}$	5 3	$\frac{a^{11}}{720}$
0 4	$\frac{a^8}{24}$	1 4	$\frac{ia^9}{24}$	2 4	$-\frac{a^{10}}{48}$	3 4	$-\frac{ia^{11}}{144}$	4 4	$\frac{a^{12}}{576}$	5 4	$\frac{ia^{13}}{2880}$
0 5	$\frac{ia^{10}}{120}$	1 5	$-\frac{a^{11}}{120}$	2 5	$-\frac{ia^{12}}{240}$	3 5	$\frac{a^{13}}{720}$	4 5	$\frac{ia^{14}}{2880}$	5 5	$-\frac{a^{15}}{14400}$

$$W(k,h+1)=\frac{-\eta i(k+1)(k+2)W(k+2,h)}{(h+1)} \tag{7}$$

and by equating the series form of Eq. 3 with the initial condition, it was obtained the initial differential transformed condition:

$$W(k,0)=\frac{(ia)^k}{k!},\ k=0,1,\ldots,n \tag{8}$$

By applying Eq. 8 into Eq. 7 some essential values of W(k, h) are obtained by iteration that are shown in Table 2.

Consequently substituting all values of W(k, h) into Eq. 4 and making the common factors:

$$C=\left(1+iax-\frac{a^2}{2}x^2-\frac{ia^3}{6}x^3+\frac{a^4}{24}x^4+\frac{ia^5}{120}x^5+\ldots\right)$$

for the elements having the same power of t, the expression of w(x, t) yields:

$$w(x,t)=C+Cia^2t+C\frac{1}{2}(ia^2)^2t^2+C\frac{1}{6}(ia^2)^3t^3+C\frac{1}{24}(ia^2)^4t^4 +C\frac{1}{120}(ia^2)^5t^5+\ldots \tag{9}$$

and so on, the rest of components W(k, h) of the iteration Eq. 7 were also obtained using the Mathematica package.

Consequently, the solution w(x, t) is written in a series form after some manipulations:

$$w(x,t)=C\left[1+(ia^2)t+\frac{(ia^2)^2}{2}t^2+\frac{(ia^2)^3}{6}t^3+\frac{(ia^2)^4}{24}t^4+\frac{(ia^2)^5}{120}t^5+\ldots\right] \tag{10}$$

The closed forms of the first and second curly brackets are identified as Taylor's series expansions of e^{iax} and e^{ia^2t}, respectively. Then the solution in a closed form is readily found to be $w_{+1}(x, t) = e^{ia(x+at)}$ (here, the subscript in w_{+1} refers to $\eta = +1$) which is the exact result. With a similar way, the values of W(k, h) in the case $\eta = -1$ are listed in Table 3.

It is clear that in the present case, there is no need to rework the solution for $\eta = -1$, because all the elements $W_{-1}(k, h)$ of a particular h are simply deduced from $W_{+1}(k, h)$ multiplied by $(-1)^h$. Hence, the formal power series solution of $w_{-1}(x, t)$ is given by:

$$\begin{aligned} w_{-1}(x,t) &= \sum_{k=0}^{n}\sum_{h=0}^{m}W_{-1}(k,h)x^kt^h \\ w_{-1}(x,t) &= \sum_{k=0}^{n}\sum_{h=0}^{m}W_{+1}(k,h)(-1)^h x^kt^h \\ &= w_{+1}(x,-t) \\ &= e^{ia(x-at)} \end{aligned} \tag{11}$$

Example 2: The one-dimensional cubic nonlinear Schrödinger equation, describing waves in optical fibers, in superconductors, in plasma and governing the evolution of any weakly nonlinear, strongly dispersive waves, etc.[31] is written in the following form:

$$iw_t+\eta w_{xx}+\varepsilon 2\,|w|^2\,w = 0 \tag{12}$$

where, $|w|^2 = ww^*$ and w^* is the conjugate of the complex valued function w(x, t) which is subjected to the initial condition: $w(x, 0) = e^{iax}$, $a\in R$. The Eq. 12 provides four types of solutions according to the values of $(\varepsilon, \eta) = (\pm 1, \pm 1)$. The parameter ε corresponds to a focusing ($\varepsilon = +1$) or defocusing ($\varepsilon = -1$) effect of the cubic nonlinearity.

The conjugate notation is used for the complex function w(x, t) and the equation under consideration is written in the following form:

$$iw_t+\eta w_{xx}+\varepsilon 2w^2w^* = 0 \tag{13}$$

As before, taking the differential transform method to the Eq. 13 together with the associated initial condition, the following relation is obtained:

$$i(h+1)\, W(k, h+1)+\eta(k+1)\,(k+2)\, W(k+2, h)+\varepsilon 2G(k, h) = 0 \tag{14}$$

where, G(k, h) is given by:

Table 3: The values of W(k, h) for η = -1

k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)
0 1	ia^2	1 1	a^3	2 1	$\frac{ia^4}{2}$	3 1	$-\frac{a^5}{6}$	4 1	$-\frac{ia^6}{24}$	5 1	$\frac{a^7}{120}$
0 2	$-\frac{a^4}{2}$	1 2	$-\frac{ia^5}{2}$	2 2	$\frac{a^6}{4}$	3 2	$\frac{ia^7}{12}$	4 2	$-\frac{a^8}{48}$	5 2	$-\frac{ia^9}{240}$
0 3	$\frac{ia^6}{6}$	1 3	$-\frac{a^7}{6}$	2 3	$-\frac{ia^8}{12}$	3 3	$\frac{a^9}{36}$	4 3	$\frac{ia^{10}}{144}$	5 3	$-\frac{a^{11}}{720}$
0 4	$\frac{a^8}{24}$	1 4	$\frac{ia^9}{24}$	2 4	$-\frac{a^{10}}{48}$	3 4	$-\frac{ia^{11}}{144}$	4 4	$\frac{a^{12}}{576}$	5 4	$\frac{ia^{13}}{2880}$
0 5	$-\frac{ia^{10}}{120}$	1 5	$\frac{a^{11}}{120}$	2 5	$\frac{ia^{12}}{240}$	3 5	$-\frac{a^{13}}{720}$	4 5	$-\frac{ia^{14}}{2880}$	5 5	$\frac{a^{15}}{14400}$

Table 4: Results of W(k, h) for η = +1, ε = +1

k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)
0 1	$i(2-a^2)$	1 1	$-a(2-a^2)$	2 1	$-\frac{ia^2}{2}(2-a^2)$	3 1	$\frac{a^3}{6}(2-a^2)$	4 1	$\frac{ia^4}{24}(2-a^2)$	5 1	$-\frac{a^5}{120}(2-a^2)$
0 2	$-\frac{1}{2}(2-a^2)^2$	1 2	$-\frac{1}{2}ia(2-a^2)^2$	2 2	$\frac{1}{4}a^2(2-a^2)^2$	3 2	$\frac{1}{12}ia^3(2-a^2)^2$	4 2	$-\frac{1}{48}a^4(2-a^2)^2$	5 2	$-\frac{1}{240}ia^5(2-a^2)^2$
0 3	$-\frac{1}{6}i(2-a^2)^3$	1 3	$\frac{1}{6}a(2-a^2)^3$	2 3	$\frac{1}{12}ia^2(2-a^2)^3$	3 3	$-\frac{1}{36}a^3(2-a^2)^3$	4 3	$-\frac{1}{144}ia^4(2-a^2)^3$	5 3	$\frac{1}{720}a^5(2-a^2)^3$
0 4	$\frac{1}{24}(2-a^2)^4$	1 4	$\frac{1}{24}ia(2-a^2)^4$	2 4	$-\frac{1}{48}a^2(2-a^2)^4$	3 4	$-\frac{1}{144}ia^3(2-a^2)^4$	4 4	$\frac{1}{576}a^4(2-a^2)^4$	5 4	$\frac{1}{2880}ia^5(2-a^2)^4$
0 5	$\frac{1}{120}i(2-a^2)^5$	1 5	$-\frac{1}{120}a(2-a^2)^5$	2 5	$-\frac{1}{240}ia^2(2-a^2)^5$	3 5	$\frac{1}{720}a^3(2-a^2)^5$	4 5	$\frac{1}{2880}ia^4(2-a^2)^5$	5 5	$-\frac{1}{14400}a^5(2-a^2)^5$

$$G(k,h)=\sum_{n=0}^{k}\sum_{m=0}^{k-n}\sum_{j=0}^{h}\sum_{l=0}^{h-j}W(n,h-j-l)\,W(m,j)\,W^*(k-n-m,l) \quad (15)$$

and by equating the series form of Eq. 3 with the initial conditions, the initial differential transformed conditions were obtained:

$$\begin{aligned} W(k,0)&=\frac{(ia)^k}{k!},\ k=0,1,\dots,n,\\ W^*(k,0)&=\frac{(-ia)^k}{k!},\ k=0,1,\dots,n \end{aligned} \quad (16)$$

rearrange Eq. 14 by writing:

$$W(k,h+1)=\frac{i\eta(k+1)(k+2)W(k+2,h)+\varepsilon 2iG(k,h)}{(h+1)} \quad (17)$$

In a similar manner as mentioned earlier, it is easier to calculate by computing the sequences {W(k, h+1)} from appropriate initial values W(k, 0):

- Case 1: η = +1, ε = +1

By applying Eq. 16 into Eq. 14, some essential values of W(k, h) are obtained by iteration that are shown in Table 4.

Consequently substituting all values of W(k, h) into Eq. 4 and making the common factors C for the elements having the same power of h, yields the following relation:

$$\begin{aligned} w(x,t)&=C+Ci(2-a^2)t+C\frac{1}{2}\left(i(2-a^2)\right)^2t^2+C\frac{1}{6}\left(i(2-a^2)\right)^3t^3\\ &+C\frac{1}{24}\left(i(2-a^2)\right)^4t^4+C\frac{1}{120}\left(i(2-a^2)\right)^5t^5+\dots \end{aligned} \quad (18)$$

thus, the following representation of the wave solution w(x, t) is obtained in a series form after some manipulations:

$$\begin{aligned} w(x,t)=&\left\{1+iax-\frac{a^2}{2}x^2-\frac{a^3}{6}ix^3+\frac{a^4}{24}x^4+\frac{a^5}{120}ix^5+\dots\right\}\\ &\times\left\{\begin{aligned}&1+i(2-a^2)t+\frac{1}{2}\left(i(2-a^2)\right)^2t^2+\frac{1}{6}\left(i(2-a^2)\right)^3t^3\\ &+\frac{1}{24}\left(i(2-a^2)\right)^4t^4+\frac{1}{120}\left(i(2-a^2)\right)^5t^5+\dots\end{aligned}\right\} \end{aligned} \quad (19)$$

The closed forms of the first and second curly brackets are identified as Taylor's series expansions of e^{iax} and $e^{i(2-a^2)t}$, respectively and the solution in a closed form is readily found to be $w(x,t)=e^{i(ax+(2-a^2)t)}$ which is the exact result:

- Case 2: η = +1, ε = -1

As above for the first case, a direct computation of the Eq. 14 yields the values of W(k, h) which are listed in Table 5.

It should be remarked that, the values of $W_{+1\pm1}(k, h)$ (where, $W_{+1\pm1}$ refers to η = 1 and ε = ±1) placed in same row can differ only by a factor:

$$\left(\frac{a^2+2}{a^2-2}\right)^h$$

fixed. This means that:

Table 5: Some results of W(k, h) for $\eta = +1$, $\varepsilon = -1$

k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)	k h	W(k, h)
0 1	$-i(a^2+2)$	1 1	$a(a^2+2)$	2 1	$\frac{1}{2}ia^2(a^2+2)$	3 1	$-\frac{1}{6}a^3(a^2+2)$	4 1	$-\frac{1}{24}ia^4(a^2+2)$	5 1	$\frac{1}{120}a^5(a^2+2)$
0 2	$-\frac{1}{2}(a^2+2)^2$	1 2	$-\frac{1}{2}ia(a^2+2)^2$	2 2	$\frac{1}{4}a^2(a^2+2)^2$	3 2	$\frac{1}{12}ia^3(a^2+2)^2$	4 2	$-\frac{1}{48}a^4(a^2+2)^2$	5 2	$-\frac{1}{240}ia^5(a^2+2)^3$
0 3	$\frac{1}{6}i(a^2+2)^3$	1 3	$-\frac{1}{6}a(a^2+2)^3$	2 3	$-\frac{1}{12}ia^2(a^2+2)^3$	3 3	$\frac{1}{36}a^3(a^2+2)^3$	4 3	$\frac{1}{144}ia^4(a^2+2)^3$	5 3	$-\frac{1}{720}a^5(a^2+2)^3$
0 4	$\frac{1}{24}(a^2+2)^4$	1 4	$\frac{1}{24}ia(a^2+2)^4$	2 4	$-\frac{1}{48}a^2(a^2+2)^4$	3 4	$-\frac{1}{144}ia^3(a^2+2)^4$	4 4	$\frac{1}{576}a^4(a^2+2)^4$	5 4	$\frac{ia^5(a^2+2)^4}{2880}$
0 5	$-\frac{1}{120}i(a^2+2)^5$	1 5	$\frac{1}{120}a(a^2+2)^5$	2 5	$\frac{1}{240}ia^2(a^2+2)^5$	3 5	$-\frac{1}{720}a^3(a^2+2)^5$	4 5	$-\frac{ia^4(a^2+2)^5}{2880}$	5 5	$\frac{a^5(a^2+2)^5}{14400}$

Table 6: Some values of $W_{\eta-}(k, h)$ for $\eta = +1$, $\varepsilon = -1$

k h	$W_{\eta+}$ (k, h)	k h	$W_{\eta+}$ (k, h)	k h	$W_{\eta+}$ (k, h)	k h	$W_{\eta+}$ (k, h)	k h	$W_{\eta+}$ (k, h)	k h	$W_{\eta+}$ (k, h)	×by
0 1	$-i(a^2+2)$	1 1	$a(a^2+2)$	2 1	$\frac{1}{2}ia^2(a^2+2)$	3 1	$-\frac{1}{6}a^3(a^2+2)$	4 1	$-\frac{1}{24}ia^4(a^2+2)$	5 1	$\frac{1}{120}a^5(a^2+2)$	$\left(\frac{a^2+2}{a^2-2}\right)$
0 2	$-\frac{1}{2}(a^2+2)^2$	1 2	$-\frac{1}{2}ia(a^2+2)^2$	2 2	$\frac{1}{4}a^2(a^2+2)^2$	3 2	$\frac{1}{12}ia^3(a^2+2)^2$	4 2	$-\frac{1}{48}a^4(a^2+2)^2$	5 2	$-\frac{1}{240}ia^5(a^2+2)^2$	$\left(\frac{a^2+2}{a^2-2}\right)^2$
0 3	$\frac{1}{6}i(a^2+2)^3$	1 3	$-\frac{1}{6}a(a^2+2)^3$	2 3	$-\frac{1}{12}ia^2(a^2+2)^3$	3 3	$\frac{1}{36}a^3(a^2+2)^3$	4 3	$\frac{1}{144}ia^4(a^2+2)^3$	5 3	$-\frac{1}{720}a^5(a^2+2)^3$	$\left(\frac{a^2+2}{a^2-2}\right)^3$
0 4	$\frac{1}{24}(a^2+2)^4$	1 4	$\frac{1}{24}ia(a^2+2)^4$	2 4	$-\frac{1}{48}a^2(a^2+2)^4$	3 4	$-\frac{1}{144}ia^3(a^2+2)^4$	4 4	$\frac{1}{576}a^4(a^2+2)^4$	5 4	$\frac{ia^5(a^2+2)^4}{2880}$	$\left(\frac{a^2+2}{a^2-2}\right)^4$
0 5	$-\frac{1}{120}i(a^2+2)^5$	1 5	$\frac{1}{120}a(a^2+2)^5$	2 5	$\frac{1}{240}ia^2(a^2+2)^5$	3 5	$-\frac{1}{720}a^3(a^2+2)^5$	4 5	$-\frac{ia^4(a^2+2)^5}{2880}$	5 5	$\frac{a^5(a^2+2)^5}{14400}$	$\left(\frac{a^2+2}{a^2-2}\right)^5$

Table 7: Some symbolic values of W(k, h) for: $\eta = -1$, $\varepsilon = \pm 1$

	$\eta = -1$, $\varepsilon = +1$	$\eta = -1$, $\varepsilon = -1$		$\eta = -1$, $\varepsilon = +1$	$\eta = -1$, $\varepsilon = -1$
k h	$W_{\eta\varepsilon}$ (k, h)	$W_{\eta\varepsilon}$ (k, h)	k h	$W_{\eta\varepsilon}$ (k, h)	$W_{\eta\varepsilon}$ (k, h)
0 1	$i(a^2+2)$	$i(a^2-2)$	1 1	$i(a^2+2)$	$-a(a^2-2)$
0 2	$-\frac{1}{2}(a^2+2)^2$	$-\frac{1}{2}(a^2-2)^2$	1 2	$-\frac{1}{2}(a^2+2)^2$	$-\frac{1}{2}ia(a^2-2)^2$

$$W_{+1-1}(k,h) = \left(\frac{a^2+2}{a^2-2}\right)^h W_{+1-1}(k,h)$$

The accompanying Table 6 shows this argument.

Now for the cases $\eta = -1$, $\varepsilon = \pm 1$ and for information, now only explicitly calculate the first few terms of W(k, h) in order to see the above underlying argument, it will not be necessary to give more terms (Table 7).

In view of these above results, the elements $W_{\eta\varepsilon}(k, h)$ are linked by a constant coefficient of the power h as:

$$\begin{aligned} W_{-1+1}(k,h) &= \left(-\frac{a^2+2}{a^2-2}\right)^h W_{+1+1}(k,h), \\ W_{-1-1}(k,h) &= (-1)^h W_{+1+1}(k,h) \end{aligned} \tag{20}$$

and the solutions read:

$$\begin{aligned} w_{-1+1}(x,t) &= \sum_{k=0}^{n}\sum_{h=0}^{m} W_{-1+1}(k,h)x^k t^h \\ &= w_{+1+1}\left(x, -\frac{a^2+2}{a^2-2}t\right) \\ &= e^{i(ax+(a^2+2)t)} \end{aligned} \tag{21}$$

and:

$$\begin{aligned} w_{-1-1}(x,t) &= \sum_{k=0}^{n}\sum_{h=0}^{m} W_{-1+1}(k,h)x^k t^h \\ &= w_{+1+1}(x,-t) \\ &= w(x,t) = e^{i(ax+(a^2-2)t)} \end{aligned} \tag{22}$$

which are similar to true results.

The wave solutions obtained in this paper, agree with the wave solutions obtained in recent papers by other methods. For the particular cases in which $\eta = 1$, a = 3 (linear case) and $\eta = 1$, $\varepsilon = \pm 1$, a = 1 (nonlinear case), our results are similar with those of variational iteration method[32]. On the other hand, Biazar and Ghazvini[33] introduced and studied the one dimensional cubic nonlinear Schrödinger equation by He's homotopy perturbation method in the form:

$$2iw_t + w_{xx} + \varepsilon 2\,|w|^2\,w = 0 \tag{23}$$

with $w(x, 0) = e^{ix}$, they obtain the solution $w(x,t) = e^{i(x+\frac{1}{2}t)}$. This equation can be converted to the Eq. 12 in the case $\eta = 1$, $\varepsilon = 1$ by the substitution $t = 2t_1$ and the solution found by the present method satisfies $w(x,t_1) = e^{i(ax+(2-a^2)t_1)}$ with a = 1, which clearly gives $w(x,t) = e^{i(x+\frac{1}{2}t)}$. The same result also obtained by the Adomian decomposition method[34].

CONCLUSION

As result, exact solutions are obtained of cited models which are useful in Mathematical Physics. It was shown that the differential transform method works well and can provide an excellent tool for exploring solutions in a quantum environment. Using some illustrative examples, it should be noted that this method is a powerful and straightforward solution method to find closed-form for linear and nonlinear Schrödinger equations. On one side, for the first case study, two exact expressions of solutions for the linear case are obtained with the iterated sequence. On the other side, for the second case study, four exact expressions of solutions for the nonlinear case have been computed exactly as in the linear case. The results have been compared to other approaches. For a future work, it is important to study the global solutions when the symbolic initial conditions are expressed by other alternative forms more complex. A significant development must be provided in this direction in order to produce the appropriate solutions. The details will be given elsewhere and will be reported in future research work. All computations are performed by using Mathematica package.

SIGNIFICANCE STATEMENTS

This study discovers new solutions describing the travelling waves that can be beneficial for the comprehension of the nonlinearity of the Schrödinger equation of a given quantum system. However, this study deals with a qualitative method that reveals novel aspects arising from nonlinear equations and produces of exact and numerical solutions. For this purpose the Mathematica package together with a specific algorithm are used to calculate particular exact solutions, this was also of great assistance in the construction of power series and for other algebraic manipulation. This study help the researchers to uncover the large possibilities to study the nonlinear Schrödinger equation introducing initial conditions with symbolic parameters that many researchers were not able to explore.

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