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Research Article

Comparative Study of Two Semi-analytical Methods for the Solution of Time-Fractional Black-Scholes Equation in a Caputo Sense

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Abstract

Background and Objectives: This research presents series solution of time-fractional Black-Scholes partial differential equation with boundary condition for a European option pricing problem in a Caputo sense. The aim of this study was to conduct the comparison of two semi-analytical methods namely the Fractional Reduced Differential Transform Method (FRDTM) and the Fractional Laplace Transform Homotopy Perturbation Method (FLTHPM) for the solution of the time-fractional Black-Scholes equation. **Materials and Methods:** These two methods are based on trans forms involving fractional derivatives. Both methods provide a closed-form solution in the form of a convergent series with easily computable components, require no restrictive assumptions. The methods are compared on time-fractional Black-Scholes equation. **Results:** The solution generated by FRDTM is in excellent agreement with that of FLTHPM. The small size of calculation in FRDTM in comparison with FLTHPM is its advantage. **Conclusion:** Hence, FRDTM is strongly recommended for the solution of time-fractional Black-Scholes equation emanating from financial market.

Key words: Black-Scholes equation, European option, homotopy perturbation method, semi-analytical method

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INTRODUCTION

The valuation of options has become an important aspect of financial engineering and mathematical finance. Black and Scholes[1] derived the most famous analytical valuation formula known as the "Black-Scholes model" for options on both dividend and non-dividend yields. The Black-Scholes model for the valuation of options is given by the following equation:

$$\frac{\partial v(x,t)}{\partial t}+\frac{\sigma^2x^2}{2}\frac{\partial^2 v(x,t)}{\partial x^2}+rx\frac{\partial v(x,t)}{\partial x}=rv(x,t) \tag{1}$$

where, $(x, t)\in R^+\times(0, T)$, v (x, t) is the price of the European call option, x is the price of the underlying asset, t is the current time, T is the time to expiry/maturity date, r is the risk neutral interest rate and σ is the volatility. Let the values of the European call and put options be denoted by $v^c(x, t)$ and $v^p(x, t)$, respectively. The payoff functions for European call and put options are given by:

$$v^c(x, t) = \text{Max}(x\text{-}E, 0) \tag{2}$$

$$v^p(x, t) = \text{Max}(E\text{-}x, 0) \tag{3}$$

where, E is the exercise price of the option. During the past few decades, many researchers have studied the existence of solutions of the Black-Scholes model using different approaches[2]. The concepts of the fractional calculus have gained much attention due to the fact that fractional differential equation provides an excellent instrument for the description of many practical dynamics phenomena emanating from applied mathematics, financial market, economics, physics and engineering[3-7]. The reduced differential transform method was introduced by Keskin and Oturanc[8] to solve both linear and non-linear Partial Differential Equations (PDEs)[9-11]. Acan *et al.*[12] applied a new local fractional reduced differential transform method for the solution of some linear and non-linear PDEs on cantor set. The main aim of this research was to compare FRDTM with the FLTHPM for the solution of time-fractional Black-Scholes equation with boundary condition for a European option pricing problem.

Definitions of terms and fractional calculus theory: This section presents some definitions of terms and fractional calculus theory.

Definition 1: The Riemann-Liouville integral operator of order $\alpha>0$ of a function $f\in C_\mu$, $\mu\geq-1$ is defined as[13]:

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,\ \alpha>0, t>0 \tag{4}$$

Where:

$$\Gamma(\alpha)=\int_0^\infty e^{-\alpha}t^{\alpha-1}dt,\ \Re(\alpha)>0$$

Definition 2: The Riemann-Liouville derivative of order $\alpha>0$ denoted by D^α is defined[13]:

$$D^\alpha f(t)=\frac{d^n}{dt^n}\left(J^{n-\alpha}f(t)\right) \tag{5}$$

Using Eq. 4, we get Eq. 5:

$$D^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}f^n(\tau)\,d\tau \tag{6}$$

where, $f\in C^n_{-1}, n\in N, n-1<\alpha\leq n$ and n is the smallest integer greater than α. Equation 6 is also called Caputo fractional derivative of $f\in C^n_{-1}$. The following result gives the properties of fractional calculus[14].

Lemma 1: If $n-1<\alpha\leq n$, $n\in N$, $f\in L^1(R)$, then the following properties hold:

$$J^\alpha\left[D^\alpha f(t)\right]=f(t)-\sum_{k=0}^n f^k(0^+)\left(\frac{t^k}{k!}\right),\ t>0 \tag{7}$$

and:

$$J^\alpha\left[D^\alpha f(t)\right]=f(t) \tag{8}$$

Definition 3: The Mittag-Leffler function denoted by $E_a(z)$ is defined as the series representation of the form:

$$E_\alpha(z)=\sum_{m=0}^\infty\frac{z^m}{\Gamma(\alpha m+1)},\alpha>0 \tag{9}$$

Equation 9 is valid in the whole complex plane.

Definition 4: The Fractional Reduced Differential Transform (FRDT) $\Phi_k(x)$ of the function $\phi(x, t)$ is defined as:

$$\Phi_k(x)=\frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}\phi(x,t)}{\partial t^{k\alpha}}\right]_{t=t_0},\ k=0,1,2,...,n,\ 0<\alpha\leq1 \tag{10}$$

Definition 5: The fractional reduced differential inverse transform of Φ_k (x) is defined as follows:

$$\phi(x,t) = \sum_{k=0}^{\infty} \Phi_k(x)(t-t_0)^{k\alpha}, 0<\alpha\leq 1 \quad (11)$$

Using Eq. 10 and 11, we state without proofs, some basic properties of FRDTM as follows:

Theorem 1: If:

$$\pi(x,t) = \phi(x,t) + \xi(x,t)$$

Then:

$$\Pi_k(x) = \Phi_k(x) + \Xi_k(x)$$

Theorem 2: If:

$$\pi(x,t) = a\varphi(x,t)$$

Then:

$$\pi_k(x,) = a\phi_k(x)$$

where, a is a constant.

Theorem 3: If:

$$\pi(x,t) = \phi(x,t)\xi(x,t)$$

Then:

$$\Pi_k(x) = \sum_{i=1}^{k} \Phi_i(x)\Xi_{k-i}(x)$$

Theorem 4: If:

$$\xi(x,t) = \frac{\partial^{n\alpha}\phi(x,t)}{\partial t^{n\alpha}}$$

Then:

$$\Xi_k(x) = \frac{\Gamma(1+(k+n)\alpha)}{\Gamma(1+k\alpha)}\Phi_{k+n}(x), n\in N$$

Theorem 5: If:

$$\phi(x,t) = \frac{x^{m\alpha}}{\Gamma(1+m\alpha)}\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$$

Then:

$$\Phi_k(x) = \frac{x^{m\alpha}}{\Gamma(1+m\alpha)}\frac{\delta_a(k-n)}{\Gamma(1+\alpha)}, \quad n,m\in N$$

Theorem 6: If:

$$\xi(x,t) = \frac{\partial^{n\alpha}\phi(x,t)}{\partial x^{n\alpha}}$$

Then:

$$\Xi_k(x) = \frac{\partial^{n\alpha}\Phi_{k+n}(x)}{\partial x^{n\alpha}}, n\in N$$

Definition 6: Let f (x) be a piece-wise continuous function on every closed interval $\{a\leq x\leq b\}\subset\{0\leq x\leq\infty\}$ there exists $f:\{0\leq x\leq\infty\}\to R$, f: $x\to f(x)$ such that $s\in R$. Then F (s) is called the Laplace transform of f (x) and is given by:

$$L(f(x))(s) := F(s) = \int_0^{\infty} f(x)e^{-sx}dx \quad (12)$$

whenever the integral exists.

Definition 7: Let L(f (x))(s) := F (s) in the transformed s-space, that is, F(s) is the Laplace transform of the function f(x), then f(x) is called the inverse Laplace transform of F (s). In that case:

$$L^{-1}(F(s)) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} F(s)\, e^{sx}ds \quad (13)$$

Definition 8: The Laplace transform of the Riemann-Liouville fractional integral is defined[15]:

$$L\,[J^{\alpha} f(x)]\,(s) := s^{-\alpha} F(s) \quad (14)$$

Definition 9: The Laplace transform of the Caputo fractional derivative is defined[15]:

$$L[D^{\alpha} f(x)](s) := s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)}f^{(k)}(0), \alpha\in(n-1,n] \quad (15)$$

IMPLEMENTATION OF THE TWO METHODS

This section presents the implementation of the fractional reduced differential transform and the fractional Laplace transform homotopy perturbation method as follows.

Application: Consider the following time-fractional Black-Scholes option pricing partial differential equation of the form[16]:

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} - \frac{\partial^2 v(x,t)}{\partial x^2} - (k-1)\frac{\partial v(x,t)}{\partial x} + kv(x,t) = 0 \quad (16)$$

Subject to the initial condition:

$$v(x,0) = \text{Max}\ (e^x\text{-}1, 0) \quad (17)$$

Notice that this system of equations contains just 2 dimensionless parameters $k=\frac{2r}{\sigma^2}$, where, k represents the balance between the rate of interests and the variability of stock returns and the dimensionless time to expiry $\frac{\sigma^2 T}{2}$, even though there are 4 dimensional parameters, E, T, σ^2 and r in the original statements of the problem.

Method of solution via FRDTM: Applying FRDT on both sides of Eq. 16 and 17 yields, respectively:

$$V_{n+1}(x)=\frac{\Gamma(1+\alpha n)}{\Gamma(1+\alpha(1+n))}\left(\frac{\partial^2 V_n(x)}{\partial x^2}+(k-1)\frac{\partial V_n(x)}{\partial x}-kV_n(x)\right) \quad (18)$$

and:

$$v_0(x) = \text{Max}(e^x\text{-}1, 0) \quad (19)$$

Therefore:

$$V_1(x)=\left(\frac{k}{\Gamma(1+\alpha)}\right)(\max(e^x,0)-\max(e^x-1,0)) \quad (20)$$

$$V_2(x)=\left(\frac{-k^2}{\Gamma(1+2\alpha)}\right)(\max(e^x,0)-\max(e^x-1,0)) \quad (21)$$

$$V_3(x)=\left(\frac{k^3}{\Gamma(1+3\alpha)}\right)(\max(e^x,0)-\max(e^x-1,0)) \quad (22)$$

In general:

$$V_n(x)=\left(\frac{(-1)^{n+1}k^n}{\Gamma(1+n\alpha)}\right)(\max(e^x,0)-\max(e^x-1,0)) \quad (23)$$

Using the fractional reduced differential inverse transform, we obtain:

$$v(x,t) = V_0(x)+V_1(x)t^a+V_2(x)t^{2a}+V_3(x)t^{3a}+... \quad (24)$$

Substituting Eq. 19-23 into Eq. 24 and solving further, one gets:

$$v(x,t)=\max(e^x-1,0)+\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}(kt^\alpha)^n}{\Gamma(1+n\alpha)}\right)(\max(e^x,0)-\max(e^x-1,0)) \quad (25)$$

Method of solution via FLTHPM: Applying the Laplace transform to Eq. 16 and 17, we obtain:

$$L[v(x,t)]=\frac{1}{s}\max(e^x-1,0)+\frac{1}{s^\alpha}L[v_{xx}+(k-1)v_x-kv] \quad (26)$$

By means of the Laplace transform inverse, Eq. 26 becomes:

$$v(x,t)=\max(e^x-1,0)+L^{-1}\left(\frac{1}{s^\alpha}L[v_{xx}+(k-1)v_x-kv]\right) \quad (27)$$

Now, we apply the homotopy perturbation method[17], one obtains:

$$\sum_{n=0}^{\infty}p^n v_n(x,t)=\max(e^x-1,0)+p\left(L^{-1}\left(\frac{1}{s^\alpha}L[\sum p^n H_n(v)]\right)\right) \quad (28)$$

where, H_n are He's polynomials. The components of He's polynomials are given by the recursive relation:

$$H_n(v) = v_{nxx}+(k\text{-}1)v_{nx}\text{-}kv_n,\ n\geq 0 \quad (29)$$

Equating the corresponding power of p on both sides in Eq. 28 yields:

$$p^0:\ v_0(x,t) = \max(e^x\text{-}1, 0) \quad (30)$$

$$p^1:\ v_1(x,t)=L^{-1}\left(\frac{1}{s^\alpha}L[H_0(v)]\right)=\frac{(-kt^\alpha)}{\Gamma(1+\alpha)}(\max(e^x-1,0)-\max(e^x,0)) \quad (31)$$

$$p^2:\ v_2(x,t)=L^{-1}\left(\frac{1}{s^\alpha}L[H_1(v)]\right)=\frac{(-kt^\alpha)^2}{\Gamma(1+2\alpha)}(\max(e^x-1,0)-\max(e^x,0)) \quad (32)$$

Continuing this manner, it is:

$$p^n:\ v_n(x,t)=L^{-1}\left(\frac{1}{s^\alpha}L[H_{n-1}(v)]\right)=\frac{(-kt^\alpha)^n}{\Gamma(1+n\alpha)}(\max(e^x-1,0)-\max(e^x,0)) \quad (33)$$

The solution of Eq. 16 subject to Eq. 17 is obtained as:

$$\begin{aligned}v(x,t)&=\lim_{p\to 1}\sum_{i=0}^{\infty}p^i v_i(x,t)\\&=\max(e^x-1,0)+\sum_{n=1}^{\infty}\left(\frac{(-kt^\alpha)^n}{\Gamma(1+n\alpha)}\right)(\max(e^x-1,0)-\max(e^x,0))\\&=\max(e^x-1,0)\,E_\alpha(-kt^\alpha)+\max(e^x,0)(1-E_\alpha(-kt^\alpha))\end{aligned} \quad (34)$$

CONCLUSION

In this research, two semi-analytical methods was used for finding the solution of time-fractional Black-Scholes equation with boundary condition for a European option pricing problem on non-dividend yield under geometric Brownian motion. The solutions generated by FRDTM and FLTHPM coincide and are in the form of convergent series with easily computable components in a direct way without using any restrictive conditions. The results show that FRDTM is found to be less computational expensive. Hence, FRDTM is a good alternative approach for the solution of time-fractional Black-Scholes equation. Some extensions and modifications of the methodology can be explored by further research. A natural extension is the applications of FRDTM and FLTHPM to higher dimensional time-fractional Black-Scholes equation for the Basket options with dividend yield under jump diffusion processes.

SIGNIFICANCE STATEMENT

This study discovers an alternative approach for the solution of time-fractional Black-Scholes equation in a Caputo sense. This study shows that the small size of calculation in FRDTM in comparison with FLTHPM is its main advantage. This study will help the researcher to uncover the critical areas of fractional calculus emanating from financial market that many researchers were not able to explore. Thus, a better solution of the time-fractional Black-Scholes like equations via FRDTM may be arrived at.

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